UNIVERSITY OF ESWATINI

FINAL EXAMINATION PAPER 2018/2019

TITLE OF PAPER :	INTRODUCTION TO STOCHASTIC PROCESSES
COURSE CODE :	STA303
TIME ALLOWED :	TWO (2) HOURS
REQUIREMENTS :	CALCULATOR AND STATISTICAL TABLES
INSTRUCTIONS :	ANSWER ANY THREE QUESTIONS

Question 1 [20 marks, 12+8]

(a) The count random variables X and Y are independent and Poisson distributed with parameters λ and μ respectively, i.e.

$$P(X=k)=\frac{\lambda^k e^{-\lambda}}{k!}, \qquad P(Y=k)=\frac{\mu^k e^{-\mu}}{k!}, \qquad k=0,1,2,\cdots,\infty.$$

Show that $Z=X+Y$ is Poisson distributed with parameter $(\lambda+\mu)$. Show also that the conditional distribution of X, given that $X+Y=n$, is binomial, and determine the parameters.

(b) It is said that a random variable X has a *Pareto distribution with parameters* x_0 *and* α ($x_0>0$ and $\alpha>0$) if X has a continuous distribution for which the p.d.f is

$$f_X(x)=\begin{cases}\frac{\alpha x_0^\alpha}{x^{\alpha+1}}, & x\geq x_0,\\ 0, & x\leq x_0.\end{cases}$$

Show that if X has this Pareto distribution, then the random variable $\log(X/x_0)$ has an exponential distribution with parameter α.

Question 2 [20 marks, 6+6+8]

(a) The count random variable X has probability generating function (PGF)

$$G_X(s)=\frac{1-s^{M+1}}{(M+1)(1-s)}$$

where M is a positive integer. Find the probability function of X.

(b) Suppose the number of car accidents in a year is Poisson distributed with parameter λ and the probabilities of an accident involving 1, 2, 3, 4 cars are 0.38, 0.55, 0.05, 0.02 respectively. Obtain an expression for the probability generating function (PGF) of X, the total number of cars involved in accidents during a year, and derive from it the expected value of X.

(c) Consider a population of individuals which can die or reproduce independently of each other with fixed generation time. Suppose the population is of size 1 initially. Let the random variable C denote the number of children of one individual where

$$\mathbf{P}(C=k)=\left(\frac{1}{2}\right)^{k-1}, \qquad k=0,1,2,\cdots$$

with PGF $G(s)$. Let the random variable X_n be the size of the n^{th} generation with PGF $G_n(s)$. Find the PGFs $G_0(s)$, $G_1(s)$, $G_2(s)$, and $G_3(s)$.

Question 3 [20 marks, 2+2+3+4+4+5]

(a) Suppose that a population of yeast cells has one individual at the start of an experiment, and each individual in the population gives birth to offspring according to a Poisson process of rate 2 per minute (with these Poisson processes being independent of one another). Let $X(t)$ denote the number of individuals in the population at time t minutes after the start of the experiment. What are the birth parameters of the birth process $\{X(t):t\geq 0\}$ in this case?

(b) Buses arrive at a bus stop according to a Poisson process of rate 4 per hour. Your answers to the following questions should be expressed in powers of e (where appropriate), but they should be simplified in all other ways.

(i) What is the probability that no bus arrives between 8:00 am and 8:30 am?

(ii) Given that no bus arrives between 8:00 am and 8:30 am, what is the probability that at least two buses arrive between 8:30 am and 9:00 am?

(iii) Each bus which arrives at the bus stop is 'out of service' with probability $1/4$, independently of all the other buses. What is the probability that at least two 'in service' buses arrive between 9:00 am and 10:00 am?

(iv) Given that exactly four buses arrive between 10:00 am and 11:00 am, what is the probability that exactly two arrived between 10:00 am and 10:20 am?

(v) Suppose I arrive at the bus stop at 11:00 am. Let T_2 denote the time (in hours) I have to wait before I have seen a total of two buses arrive. Find (with justification) the probability density function of T_2.

Question 4 **[20 marks, 3+1+6+10]**

(a) Let $\{X_n : n = 0, 1, 2, \cdots\}$ be a Markov chain with state space $\{1, 2, 3, 4, 5\}$ and transition matrix

$$\begin{pmatrix} 0 & \frac{1}{4} & \frac{1}{8} & \frac{1}{4} & \frac{3}{8} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & \frac{1}{4} & \frac{3}{8} & \frac{1}{4} & \frac{1}{8} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

(i) Draw a transition graph for this Markov chain.

(ii) List the absorbing states of the Markov chain.

(iii) Using first-step analysis, or otherwise, find the probability that the Markov chain is eventually absorbed in state 5, given that $X_0 = 1$. Show your working.

(b) Consider the following transition matrix with state space $E = \{1, 2, 3, 4\}$

$$\mathbf{P} = \begin{pmatrix} q & p & 0 & 0 \\ q & 0 & p & 0 \\ 0 & q & 0 & p \\ 0 & 0 & q & p \end{pmatrix}.$$

Show that this chain is irreducible (all states communicate) and ergodic. Find the long run distribution.

Question 5 **[20 marks, 8+4+4+4]**

(a) An insurance company offers annual motor-car insurance based on a "no claims discount" system with levels of discount 0%, 30% and 60%. A policyholder who makes no claims during the year either moves to the next higher level of discount or remains at the top level. If there is exactly one claim during the year, the policyholder either moves down one level or stays at the bottom level (0%). If there is more than one claim during the year, the policyholder either moves down to or stays at the bottom level. For a particular policyholder, it may be assumed that claims arise in a Poisson process at rate $\lambda > 0$. Explain why the situation described above is suitable for modelling in terms of a Markov chain with three states, and write down the transition probability matrix in terms of λ.

(b) Suppose you have two computer monitors (independently) one in your office having lifetime X_1 exponential with rate $\lambda_1 = 0.25$ (hence mean = 4 years), and the other at home having lifetime X_2 exponential with $\lambda_2 = 0.5$ (hence mean = 2 years). As soon as one of them breaks, you must order a new monitor.

(i) What is the expected amount of time until you need to order a new monitor?

(ii) What is the probability that the office monitor is the first to break?

(iii) Given that the office monitor broke first, what was the expected lifetime of the monitor?