UNIVERSITY OF ESWATINI

SUPPLEMENTARY EXAMINATION PAPER 2018/2019

TITLE OF PAPER : DISTRIBUTION THEORY

COURSE CODE : ST301

TIME ALLOWED : TWO (2) HOURS

REQUIREMENTS : CALCULATOR

INSTRUCTIONS : ANSWER ANY THREE QUESTIONS

Question 1 **[20 marks, 4+6+4+6]**

(a) The joint density of (X, Y) is

$$f_{X,Y}(x,y) = \begin{cases} x^k e^{-ky}, & 0 < x < y < \infty, \\ 0, & \text{otherwise.} \end{cases}$$

(i) Verify that this is a valid density when $k = 1$.

(ii) Find $f_X(x)$, $f_Y(y)$ and $f_{Y|X}(y|x)$, representing the marginal density of X, Y and the conditional density of Y given $X = x$ respectively.

(b) Let $X \sim$ Bernuolli(p), that is, $P(X = x) = p^x(1-p)^{1-x}$, $\quad x = 0, 1$. Find the moment generating function of X. Let $Y = NX$, when N Poisson(μ) with

$$P(N = n) = \frac{\mu^n e^{-\mu}}{n!} \qquad n = 0, 1, \cdots,$$

and N is independent of X. Derive the moment generating function of Y.

Question 2 **[20 marks, 3+3+4+2+4+4]**

(a) You have a coin and two dice. The coin is biased and comes up head with probability p and tail with probability $q = 1 - p$. Dice 1 (D_1) is a fair dice. Dice 2 (D_2) is a biased dice with

$$P(D_2 = 1) = P(D_2 = 6) = \frac{1}{12}$$
$$P(D_2 = 2) = P(D_2 = 5) = \frac{1}{6}$$
$$P(D_2 = 3) = P(D_2 = 4) = \frac{1}{4}$$

You flip the biased coin. If it comes up a head, you throw Dice 1. Otherwise you throw Dice 2. Let X denote the result of the dice throw.

(i) What is the probability that the outcome of the throw is 6?

(ii) Given the outcome is 6, what is the probability that the dice thrown was dice 1?

(iii) After the first throw of dice, you throw the dice you have just thrown again. What is the probability that the two throws add up to 3?

(b) Let Z be a random variable with density

$$f_Z(z) = \frac{1}{2}e^{-|z|}, \qquad \text{for } 0 < z < \infty.$$

(i) Show that f_Z is a valid density.

(ii) Find the moment generating function of Z and specify the interval where the MGF is well-defined.

(iii) By considering the cumulant generating function or otherwise, evaluate $E(Z)$ and $Var(Z)$.

Question 3 [20 marks, 3+4+3+4+6]

(a) Without a vaccine, the probability of contracting disease D_1 is 0.2, while for disease D_2 it is 0.05. An individual will not contract both diseases at the same time.

Vaccine A lowers the probability of contracting disease D_1 to 0.1, and disease D_2 to 0.02. Vaccine B lowers these probabilities to 0.05 and 0.04, respectively.

The proportion of the population who have not received either vaccine is 0.1, the proportion who have been vaccinated by vaccine A is 0.4, and the proportion vaccinated by vaccine B is 0.5.

(i) What is the probability that a particular patient has developed disease D_1?

(ii) Given a patient has developed disease D_1, what is the probability that he/she has been vaccinated with either vaccine A or B?

(b) Without a vaccine, the death rate is 0.1 for a patient with disease D_1, and 0.5 for a patient with disease D_2. If a patient receives vaccine A and still develops a disease, the death rate is 0.06 for disease D_1 and 0.1 for disease D_2. If a patient receives vaccine B and still develops a disease, the corresponding death rates are 0.01 and 0.2, respectively.

(i) Show that for any events A, B and C, we have

$$P(A \cap B \cap C) = P(C|A \cap B)P(B|A)P(A)$$

(ii) Find the probability that a particular individual satisfies the conditions that he/she is not vaccinated, develops disease D_2, and dies eventually.

(iii) Given that a patient developed disease D_2 and died, find the probability that the patient has not been vaccinated.

Question 4 [20 marks, 5+5+5+5]

(a) Let $N(t)$ be the number of telephone calls at an exchange in the interval $(0, t]$ We suppose that $\{N(t), t \geq 0\}$ is a Poisson process with rate $\lambda = 10$ per hour. Calculate the probability that no calls will be received during each of two consecutive 15-minute periods.

(b) Let $N(t)$ be the number of failures in the interval $(0, t]$. We suppose that $\{N(t), t \geq 0\}$ is a Poisson process with rate $\lambda = 1$ per week. Calculate the probability that

(i) the system operates without failure during two consecutive weeks,

(ii) the system will have exactly two failures during a given week, knowing that it operated without failure during the previous two weeks,

(iii) less than two weeks elapse before the third failure occurs

Question 5 [20 marks, 6+6+8]

(a) The annual number of hurricanes forming in the Atlantic basin has a Poisson distribution with parameter λ. Each hurricane that forms has probability p of making landfall independent of all other hurricanes. Let X be the number of hurricanes that form in the basin and Y be the number that make landfall. Find:

(i) $E(Y)$,

(ii) $\text{Corr}(X, Y)$.

(b) Let X be such that the distribution of X given $Y = y$ is Poisson, parameter y. Let $Y \sim \text{Poisson}(\mu)$. Show that

$$G_{X+Y}(s) = \exp\left\{\mu(se^{s-1} - 1)\right\}$$