

**UNIVERSITY OF SWAZILAND**



**SUPPLEMENTARY EXAMINATION PAPER 2018**

**TITLE OF PAPER :       PROBABILITY THEORY**

**COURSE CODE        :       ST. 201**

**TIME ALLOWED       :       3 HOURS**

**INSTRUCTIONS       :       ANSWER ANY FIVE QUESTIONS.**

**REQUIREMENTS     :       SCIENTIFIC CALCULATOR**

### Question 1

- a) If  $P(A) = 0.25$  and  $P(B) = 0.8$ , show that  $0.05 \leq P(A \cap B) \leq 0.25$ .  
(5 Marks)
- b) Let A and B be Events in a sample space  $\Omega$  such that  $P(A) = \frac{1}{2} = P(B)$  and  $P(A^c \cap B^c) = \frac{1}{3}$ . Find  $P(A \cup B^c)$ .  
(5 Marks)
- c) A box of fuses contains 20 fuses, of which 5 are defective. If 3 of the fuses are selected at random and removed from the box in succession without replacement, what is the probability that all three fuses are defective?  
(5 Marks)
- d) Suppose box A contains 4 red and 5 blue chips and box B contains 6 red and 3 blue chips. A chip is chosen at random from the box A and placed in box B. Finally, a chip is chosen at random from among those now in box B. What is the probability a blue chip was transferred from box A to box B given that the chip chosen from box B is red?  
(5 Marks)

### Question 2

A continuous random variable X has cumulative distribution function:

$$F_X(x) = \begin{cases} 0, & \text{if } x \leq 0 \\ \sqrt{x}, & \text{if } 0 < x \leq 1 \\ 1, & \text{if } x > 1 \end{cases}$$

- a) Find the probability density function of X.  
(4 Marks)
- b) Calculate the expectation and variance of X.  
(12 Marks)
- c) Calculate the lower quartile of X.  
(4 Marks)

### Question 3

- a) The random variable X is uniformly distributed on the interval (0, 1). Derive the PDF of the random variable  $Y = -\ln X$ .  
(10 Marks)
- b) Consider two independent random variables  $X_1$  and  $X_2$ , distributed exponentially with  $\lambda = 1$ . That is,

$$f_X(x) = \begin{cases} e^{-x}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

Calculate the PDF of  $X_1 + X_2$ .

(10 Marks)

**Question 4**

If the joint moment generating function of the random variable X and Y is

$$M(s, t) = \exp(s + 3t + 2s^2 + 18t^2 + 12st)$$

What is the Covariance of X and Y?

(20 Marks)

**Question 5**

a) Let X and Y be random variables such that X has density function

$$f_X(x) = 24x^2, \quad 0 < x < \frac{1}{2}$$

and the conditional density of Y given X = x is

$$p(y|x) = \frac{y}{2x^2}, \quad 0 < y < 2x$$

What is the conditional density of X given Y = y over the appropriate domain?

(10 Marks)

b) Let the joint density of two random variables x and y be given by

$$f(x, y) = \frac{1}{6}(x + 4y), \quad 0 < x < 2, 0 < y < 1$$

Find the probability of  $X \leq 1$  given that  $y = \frac{1}{2}$ .

(10 Marks)

**Question 6**

a) Let X and Y be discrete random variables with joint density

$$p(x, y) = \frac{x + 2y}{18}, \quad x = 1, 2; y = 1, 2$$

What is the covariance  $\sigma_{XY}$  between X and Y.

(15 Marks)

b) If  $Var(X + Y) = 3$ ,  $Var(X - Y) = 1$ ,  $E(X) = 1$ , and  $E(Y) = 2$ , the what is  $E(XY)$ ?

(5 Marks)

**Question 7**

a) Let X and Y be discrete random variables with joint probability mass function

$$p(x, y) = \frac{1}{21}(x + y), \quad x = 1, 2, 3; y = 1, 2$$

What is the conditional mean of  $X$  given  $Y=y$ , that is  $E(X|y)$ ?

(10 Marks)

- b) Let  $X$  and  $Y$  be continuous random variables with joint probability density function

$$f(x, y) = e^{-y}, \quad 0 < x < y < \infty$$

What is the conditional variance of  $Y$  given that  $X = x$ ?

(10 Marks)

### **Question 8**

- a) Let each of the independent random variables  $X$  and  $Y$  have the density function

$$f(x) = e^{-x}, \quad 0 < x < \infty$$

What is the joint density of  $U = X$  and  $V = 2X + 3Y$  and the domain on which this density is positive?

(10 Marks)

- b) Let  $X$  and  $Y$  be independent random variables, each with density function

$$f(x) = \lambda e^{-\lambda x}, \quad 0 < x < \infty$$

where  $\lambda > 0$ . Let  $U = X + 2Y$  and  $V = 2X + Y$ . What is the joint density of  $U$  and  $V$ ?

(10 Marks)