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UNIVERSITY OF SWAZILAND

MAIN EXAMINATION PAPER 2017

TITLE OF PAPER	: TOPICS IN STATISTICS (STATISTICAL MODELLING)
COURSE CODE	: ST 405
TIME ALLOWED	: THREE (3) HOURS
REQUIREMENTS	: CALCULATOR AND STATISTICAL TABLES
INSTRUCTIONS	: ANSWER ANY FIVE QUESTIONS

Question 1

In a prospective, randomized study to investigate the capacity of aspirin to prevent pregnancy-induced hypertension (=high blood pressure), 65 women were treated with a daily dose of either aspirin (34 women) or placebo (31 women) during the third trimester of pregnancy. The result is summarized in the Table 1 below;

Table 1: Contingency table for study comparing Aspirin and Placebo

	Hypertension	No hypertension	Total
Aspirin treatment	4	30	34
Placebo treatment	11	20	31
Total	15	50	65

In Table 2, the probability that the test statistic n_{11} from Fisher's exact test attains a specific value t is given (n_{11} = number of women treated with Aspirin with hypertension).

Table 2: Conditional probability that n_{11} attains value t for data in Table 1

t	0	1	2	3	4	5	6	7
$P_{H_0}(n_{11}=t\,\vert\ldots$	0	0	0.001	0.004	0.019	0.060	0.131	0.205

t	8	9	10	11	12	13	14	15
$P_{H_0}(n_{11}=t\,\vert\ldots$	0.230	0.186	0.107	0.043	0.012	0.002	0	0

a) Fisher's exact test evaluates the probability $P_{H_0}(n_{11}=t\,\vert\ldots)$ under a certain condition. What is this condition (what are the dots in this expression)? How are $P_{H_0}(n_{11}=t\,\vert\ldots)$ in Table 2 calculated? State a formula for this probability.

(2 p)

b) Based on Table 2, calculate the one-sided p-value for Fisher's exact test for testing the null hypothesis that treatment and hypertension are independent versus the alternative that aspirin has a hypertension preventing effect. Interpret the result.

(3 p)

c) Consider a two-sided test for the independence null hypothesis versus the alternative that aspirin has any (positive or negative) effect on hypertension for pregnant women. There are diffierent methods to define a two-sided p-value based on Fisher's exact test. A simple but uncommon way is to multiply the one-sided p-value by 2. Choose another method and describe exactly how the two-sided p-value is defined. Compute the two-sided p-value and interpret.

(3 p)

d) Considering again the one-sided test problem: Calculate the mid p-value and mention an advantage and a disadvantage of using mid p-values in contrast to usual p-values.

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Question 2

Consider the data in Question 1 but apply now large sample inference.

a) Estimate the odds ratio θ and interpret it.

(3 p)

b) Construct an approximate 95% CI for the odds ratio (Hint: an asymptotic expression for the variance of $log(\hat{\theta})$ is $\frac{1}{n_{11}}+\frac{1}{n_{12}}+\frac{1}{n_{13}}+\frac{1}{n_{14}}$.

(4 p)

c) Test the null hypothesis of independence versus the one-sided alternative that aspirin has a hypertension preventing effect.

(3 p)

d) Use a test of your choice which uses a large sample distribution for the test statistic and report whether the two-sided test is significant at the 5% level.

(3 p)

Question 3

Let $Y_1, \ldots, Y_n$ be independent random variables with binary distribution,

$$P(Y_i = 1) = p_i \quad P(Y_i = 0) = 1 - p_i$$

a) We use a generalized linear model with identity link function $g(x) = x$ and we assume that the probability p_i for the i-th observation to be 1 can be expressed as linear function depending on a explanatory variable x_i:

$$g\left(E(Y_i)\right) = E(Y_i) = \eta_i = \alpha + \beta x_i$$

(i) Determine the log likelihood function as function of the parameters α and β.

(5 Marks)

(ii) Compute the likelihood equations for the ML-estimation of the parameters α and β. Mention how these equations are used to calculate the ML-estimates.

(5 marks)

b) The random variable y has a distribution in the exponential family, if its *p.d.f(or p.m.f)* can be written as

$$f(y;\theta,\phi) = exp\left\{\frac{y\theta - b(\theta)}{a(\phi)} + c(y,\phi)\right\}$$

for a specific function $a(\phi)$, $b(\theta)$, and $c(y,\phi)$, where $a(\phi) > 0$ and $b(\theta)$ has up to twice derivatives. Let $l(\theta,\phi;y)$ denote associated log-likelihood function. It is known that $E\left(\frac{\delta l}{\delta\theta}\right) = 0$ and $Var\left(\frac{\delta l}{\delta\theta}\right) = -E\left(\frac{\delta^2 l}{\delta\theta^2}\right)$. Show that $E(Y) = b'(\theta)$ and $Var(Y) = a(\phi)b''(\theta)$.

(10 Marks)

Question 4

Suppose we have the following 2×2 table with all n_{ij}positive:

	Y		
X	Yes	No	Total
Yes	n_{11}	n_{12}	n_{1+}
No	n_{21}	n_{22}	n_{2+}
Total	n_{+1}	n_{+2}	n

a) When X and Y are independent, given n, n_{1+}, n_{+1}, we have

$$p(n_{11}|n, n_{1+}, n_{+1}) = \frac{n_{1+}!\, n_{+1}!\, n_{2+}!\, n_{+2}!}{n!\, n_{11}!\, n_{12}!\, n_{21}!\, n_{22}!}$$

Show that

$$E(n_{11}|n, n_{1+}, n_{+1}) = \frac{n_{1+}\, n_{+1}}{n}$$

(12 Marks)

b) Show that the sample odds ratio can be expressed as $\hat{\theta} = \frac{n_{11}n_{22}}{n_{12}n_{21}}$.

(8 Marks)

Question 5

An experiment was conducted to assess the efficacy of two species of insect predators (melanarius, tachyporus) in controlling aphids (an insect pest) on plants grown in separate insect-proof cages. Observations were made over eight consecutive week-long periods, with fresh plants, aphid populations and predators being introduced each week. Eight plants were observed each week, with two plants receiving each of four treatment combinations (no predators, melanarius only, tachyporus only, both predators). In four of the eight weeks a fungicide was also applied to all the plants, to control outbreaks of mildew. It was expected that there might be differences in the development of aphid populations and the efficacy of the predators between the week-long periods because of variation in the temperature as well as the use of different batches of predators and aphids.

At the end of each week period, a count of the number of aphids was made for each plant. The output APPENDIX A shows the results of analysing the counts of aphids at the end of each week using a generalised linear model which allows overdispersion to be detected if it is present. The application of the fungicide to all the plants in particular weeks potentially imposes a more complex design structure, but a simplified structure was assumed to allow analysis using a generalised linear model. The fitted model allows for the main effects of each predator and the fungicide treatment, and for the interactions between these effects. The fitted model also includes terms accounting for differences between periods after allowing for fungicide effects, and differences in the separate and combined effects of the two predators between periods (again, after allowing for fungicide effects), these potentially contributing extra variability beyond the plant-to-plant variability against which treatment effects should be assessed.

a) Identify the components of a generalised linear model (GLM), showing how the expected value of the response is related to the explanatory model. Describe the particular form of GLM that is appropriate for these data.

(5 Marks)

b) Briefly describe the method of iterative reweighted least squares used in fitting a generalised linear model.

(4 Marks)

c) Interpret the results of the analysis, in particular identifying the separate and combined efficacies of the two insect predators, and the impact of the fungicide application.

(8 Marks)

d) Describe what is meant by overdispersion, and indicate how evidence for overdispersion can be identified and tested for using the analysis output.

(3 Marks)

Question 6

a) Define the survivor function $S(t)$ and the hazard function $h(t)$ for a continuous random variable T measuring lifetime. Write down an expression for the survivor function in terms of the hazard function.

(3 Marks)

b) The exponential distribution has constant hazard function $h(t) = \lambda$.Write down expressions for the density of the exponential distribution and the mean of this distribution in terms of λ.

(2 Marks)

c) Explain what is meant by a right-censored observation. Give two different examples of ways in which a right-censored observation might arise.

(3 Marks)

d) After a radical mastectomy for breast cancer, ten female patients were randomly assigned to one of two groups, an experimental group who received chemotherapy, and a control group who received no drugs. At the end of two years, survival times in months were recorded and are given in the table below. A right-censored observation is denoted by +, so 16+ denotes a right-censored observation at 16 months.

Experimental group	23	16+	18+	20+	24+
Control group	15	18	19	19	20

Compute the Kaplan-Meier estimate of the survivor function for each group and plot the results on one graph.

(10 Marks)

e) If survival times for the control group have an exponential distribution, estimate the hazard rate.

(2 Marks)

Question 7

Consider the following experiment on visual perception using random-dot stereograms. A random-dot stereogram is composed of two rectangles placed side by side, where each rectangle appears to consist only of randomly scattered dots, without any image. When viewed with only one eye functioning, the viewer cannot see a hidden image. However, when viewed with both eyes, if a person focuses the eyes in front of or behind the pair of images, then a three-dimensional hidden image of a diamond can be seen. Sometimes the diamond image can be seen quickly, but on other occasions it can take a while before it can be perceived. Here the response variable is the time in seconds needed to perceive the diamond image. The experiment investigated whether giving a person prior knowledge about the shape of the image reduces the time needed to recognise it. Forty-three subjects (group NV) received just verbal information about the shape of the hidden object. Thirty-five subjects (group VV) received both verbal information and visual information, for example a drawing of the hidden object.

a) The response time T to perceive the diamond image has a Weibull distribution with hazard function

$$h(t) = \lambda \gamma t^{\gamma - 1} \quad , \; t \geq 0$$

where $\lambda > 0$ and $\gamma > 0$are parameters to be estimated. Show that the parameters are given by the intercept and slope of the theoretical relationship of the logarithm of the cumulative hazard function plotted against the logarithm of time.

(4 Marks)

b) Write down the proportional hazards model for a vector of p time-constant covariates, X, assuming a Weibull baseline hazard function. Interpret each term in your model. Write down an expression for the hazard ratio for this model.

(4 Marks)

c) Data from the random-dot stereogram experiment were analysed using a proportional hazards model with a Weibull baseline hazard function. One explanatory variable was included in the model: GroupVV, taking the value 1 if the subject was in Group VV, or 0 if the subject was in Group NV. The following edited computer output shows the results from the fitted model.

	Estimate	Standard Error
λ	0.060	0.019
γ	1.260	0.104
Group VV	0.552	0.233

(i) Use the fitted model to estimate the hazard ratio for a subject in the VV Group compared to a subject in the NV Group. Construct a 95% confidence interval for this hazard ratio. Which group, on average, has the shorter response times?

(4 Marks)

(ii) What is the estimated hazard function for a subject in Group NV? What is the estimated hazard function for a subject in Group VV? How does the estimated hazard function change with time for each group?

(4 Marks)

d) Someone suggested using an exponential distribution, instead of a Weibull distribution, to model the response times. What advice should you give regarding this idea? Justify your answer.

(4 Marks)

APPENDIX (Output for Question 5)

Accumulated analysis of deviance

Change	d.f.	deviance	mean deviance	deviance ratio	*p*-value
+ fungicide	1	23.91	23.91	1.88	0.179
+ tachyporus	1	990.58	990.58	78.08	<.001
+ melanarius	1	0.88	0.88	0.07	0.794
+ fungicide.period	6	1116.41	186.07	14.67	<.001
+ fungicide.tachyporus	1	183.47	183.47	14.46	<.001
+ fungicide.melanarius	1	4.63	4.63	0.36	0.550
+ tachyporus.melanarius	1	3.13	3.13	0.25	0.623
+ fungicide.tachyporus.period	6	516.48	86.08	6.79	<.001
+ fungicide.melanarius.period	6	108.37	18.06	1.42	0.236
+ fungicide.tachyporus.melanarius	1	0.01	0.01	0.00	0.975
+ fungicide.tachyporus.melanarius.period	6	197.20	32.87	2.59	0.037
Residual	32	405.97	12.69		
Total	63	3551.05	56.37		

Dispersion parameter is estimated to be 12.7 from the residual deviance
The parameters estimate the effect of applying the fungicide or each predator

Estimates of some parameters

Parameter	estimate	s.e.	t value	*p*-value	antilog of estimate
Constant	5.218	0.185	28.14	<.001	184.5
fungicide yes	–0.455	0.298	–1.53	0.136	0.6341
tachyporus yes	–1.140	0.377	–3.03	0.005	0.3198
melanarius yes	0.108	0.255	0.42	0.676	1.114
fungicide yes.tachyporus yes	0.092	0.592	0.15	0.878	1.096
fungicide yes.melanarius yes	–0.099	0.416	–0.24	0.813	0.9055
tachyporus yes.melanarius yes	0.425	0.486	0.87	0.388	1.529
fungicide yes.tachyporus yes.melanarius yes	0.458	0.750	0.61	0.545	1.581

(The s.e. values are based on the residual deviance)

Predicted means (standard errors)

		melanarius			
		No		Yes	
tachyporus	No	128.34	(14.15)	131.19	(14.57)
	Yes	43.81	(8.18)	69.63	(11.27)

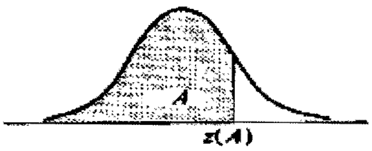
		tachyporus			
		No		Yes	
fungicide	No	97.00	(11.11)	13.03	(2.85)
	Yes	80.66	(9.72)	42.66	(6.96)

		melanarius			
		No		Yes	
fungicide	No	54.78	(8.16)	55.25	(8.06)
	Yes	72.62	(9.71)	50.69	(6.97)

Normal Distribution

Table C-1. Cumulative Probabilities of the Standard Normal Distribution.

Entry is area A under the standard normal curve from $-\infty$ to $z(A)$

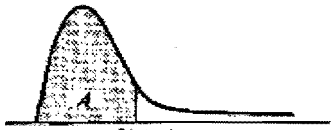


z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998

Chi-Square Distribution

Table C-2. Percentiles of the χ^2 Distribution

Entry is $\chi^2(A;\nu)$ where $P\{\chi^2(\nu) \le \chi^2(A;\nu)\} = A$



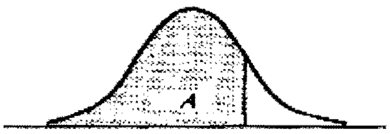
$\chi^2(A;\nu)$

	A									
ν	.005	.010	.025	.050	.100	.900	.950	.975	.990	.995
1	0.0^4393	0.0^3157	0.0^3982	0.0^2393	0.0158	2.71	3.84	5.02	6.63	7.88
2	0.0100	0.0201	0.0506	0.103	0.211	4.61	5.99	7.38	9.21	10.60
3	0.072	0.115	0.216	0.352	0.584	6.25	7.81	9.35	11.34	12.84
4	0.207	0.297	0.484	0.711	1.064	7.78	9.49	11.14	13.28	14.86
5	0.412	0.554	0.831	1.145	1.61	9.24	11.07	12.83	15.09	16.75
6	0.676	0.872	1.24	1.64	2.20	10.64	12.59	14.45	16.81	18.55
7	0.989	1.24	1.69	2.17	2.83	12.02	14.07	16.01	18.48	20.28
8	1.34	1.65	2.18	2.73	3.49	13.36	15.51	17.53	20.09	21.96
9	1.73	2.09	2.70	3.33	4.17	14.68	16.92	19.02	21.67	23.59
10	2.16	2.56	3.25	3.94	4.87	15.99	18.31	20.48	23.21	25.19
11	2.60	3.05	3.82	4.57	5.58	17.28	19.68	21.92	24.73	26.76
12	3.07	3.57	4.40	5.23	6.30	18.55	21.03	23.34	26.22	28.30
13	3.57	4.11	5.01	5.89	7.04	19.81	22.36	24.74	27.69	29.82
14	4.07	4.66	5.63	6.57	7.79	21.06	23.68	26.12	29.14	31.32
15	4.60	5.23	6.26	7.26	8.55	22.31	25.00	27.49	30.58	32.80
16	5.14	5.81	6.91	7.96	9.31	23.54	26.30	28.85	32.00	34.27
17	5.70	6.41	7.56	8.67	10.09	24.77	27.59	30.19	33.41	35.72
18	6.26	7.01	8.23	9.39	10.86	25.99	28.87	31.53	34.81	37.16
19	6.84	7.63	8.91	10.12	11.65	27.20	30.14	32.85	36.19	38.58
20	7.43	8.26	9.59	10.85	12.44	28.41	31.41	34.17	37.57	40.00
21	8.03	8.90	10.28	11.59	13.24	29.62	32.67	35.48	38.93	41.40
22	8.64	9.54	10.98	12.34	14.04	30.81	33.92	36.78	40.29	42.80
23	9.26	10.20	11.69	13.09	14.85	32.01	35.17	38.08	41.64	44.18
24	9.89	10.86	12.40	13.85	15.66	33.20	36.42	39.36	42.98	45.56
25	10.52	11.52	13.12	14.61	16.47	34.38	37.65	40.65	44.31	46.93
26	11.16	12.20	13.84	15.38	17.29	35.56	38.89	41.92	45.64	48.29
27	11.81	12.88	14.57	16.15	18.11	36.74	40.11	43.19	46.96	49.64
28	12.46	13.56	15.31	16.93	18.94	37.92	41.34	44.46	48.28	50.99
29	13.12	14.26	16.05	17.71	19.77	39.09	42.56	45.72	49.59	52.34
30	13.79	14.95	16.79	18.49	20.60	40.26	43.77	46.98	50.89	53.67
40	20.71	22.16	24.43	26.51	29.05	51.81	55.76	59.34	63.69	66.77
50	27.99	29.71	32.36	34.76	37.69	63.17	67.50	71.42	76.15	79.49
60	35.53	37.48	40.48	43.19	46.46	74.40	79.08	83.30	88.38	91.95
70	43.28	45.44	48.76	51.74	55.33	85.53	90.53	95.02	100.4	104.2
80	51.17	53.54	57.15	60.39	64.28	96.58	101.9	106.6	112.3	116.3
90	59.20	61.75	65.65	69.13	73.29	107.6	113.1	118.1	124.1	128.3
100	67.33	70.06	74.22	77.93	82.36	118.5	124.3	129.6	135.8	140.2

Student's Distribution (t Distribution)

Table C-4 Percentiles of the t Distribution

Entry is $t(A; \nu)$ where $P\{t(\nu) \leq t(A; \nu)\} = A$



$t(A; \nu)$

ν	A = .60	.70	.80	.85	.90	.95	.975
1	0.325	0.727	1.376	1.963	3.078	6.314	12.706
2	0.289	0.617	1.061	1.386	1.886	2.920	4.303
3	0.277	0.584	0.978	1.250	1.638	2.353	3.182
4	0.271	0.569	0.941	1.190	1.533	2.132	2.776
5	0.267	0.559	0.920	1.156	1.476	2.015	2.571
6	0.265	0.553	0.906	1.134	1.440	1.943	2.447
7	0.263	0.549	0.896	1.119	1.415	1.895	2.365
8	0.262	0.546	0.889	1.108	1.397	1.860	2.306
9	0.261	0.543	0.883	1.100	1.383	1.833	2.262
10	0.260	0.542	0.879	1.093	1.372	1.812	2.228
11	0.260	0.540	0.876	1.088	1.363	1.796	2.201
12	0.259	0.539	0.873	1.083	1.356	1.782	2.179
13	0.259	0.537	0.870	1.079	1.350	1.771	2.160
14	0.258	0.537	0.868	1.076	1.345	1.761	2.145
15	0.258	0.536	0.866	1.074	1.341	1.753	2.131
16	0.258	0.535	0.865	1.071	1.337	1.746	2.120
17	0.257	0.534	0.863	1.069	1.333	1.740	2.110
18	0.257	0.534	0.862	1.067	1.330	1.734	2.101
19	0.257	0.533	0.861	1.066	1.328	1.729	2.093
20	0.257	0.533	0.860	1.064	1.325	1.725	2.086
21	0.257	0.532	0.859	1.063	1.323	1.721	2.080
22	0.256	0.532	0.858	1.061	1.321	1.717	2.074
23	0.256	0.532	0.858	1.060	1.319	1.714	2.069
24	0.256	0.531	0.857	1.059	1.318	1.711	2.064
25	0.256	0.531	0.856	1.058	1.316	1.708	2.060
26	0.256	0.531	0.856	1.058	1.315	1.706	2.056
27	0.256	0.531	0.855	1.057	1.314	1.703	2.052
28	0.256	0.530	0.855	1.056	1.313	1.701	2.048
29	0.256	0.530	0.854	1.055	1.311	1.699	2.045
30	0.256	0.530	0.854	1.055	1.310	1.697	2.042
40	0.255	0.529	0.851	1.050	1.303	1.684	2.021
60	0.254	0.527	0.848	1.045	1.296	1.671	2.000
120	0.254	0.526	0.845	1.041	1.289	1.658	1.980
∞	0.253	0.524	0.842	1.036	1.282	1.645	1.960

Table C-4 (Continued) Percentiles of the *t* Distribution

	A						
ν	.98	.985	.99	.9925	.995	.9975	.9995
1	15.895	21.205	31.821	42.434	63.657	127.322	636.590
2	4.849	5.643	6.965	8.073	9.925	14.089	31.598
3	3.482	3.896	4.541	5.047	5.841	7.453	12.924
4	2.999	3.298	3.747	4.088	4.604	5.598	8.610
5	2.757	3.003	3.365	3.634	4.032	4.773	6.869
6	2.612	2.829	3.143	3.372	3.707	4.317	5.959
7	2.517	2.715	2.998	3.203	3.499	4.029	5.408
8	2.449	2.634	2.896	3.085	3.355	3.833	5.041
9	2.398	2.574	2.821	2.998	3.250	3.690	4.781
10	2.359	2.527	2.764	2.932	3.169	3.581	4.587
11	2.328	2.491	2.718	2.879	3.106	3.497	4.437
12	2.303	2.461	2.681	2.836	3.055	3.428	4.318
13	2.282	2.436	2.650	2.801	3.012	3.372	4.221
14	2.264	2.415	2.624	2.771	2.977	3.326	4.140
15	2.249	2.397	2.602	2.746	2.947	3.286	4.073
16	2.235	2.382	2.583	2.724	2.921	3.252	4.015
17	2.224	2.368	2.567	2.706	2.898	3.222	3.965
18	2.214	2.356	2.552	2.689	2.878	3.197	3.922
19	2.205	2.346	2.539	2.674	2.861	3.174	3.883
20	2.197	2.336	2.528	2.661	2.845	3.153	3.849
21	2.189	2.328	2.518	2.649	2.831	3.135	3.819
22	2.183	2.320	2.508	2.639	2.819	3.119	3.792
23	2.177	2.313	2.500	2.629	2.807	3.104	3.768
24	2.172	2.307	2.492	2.620	2.797	3.091	3.745
25	2.167	2.301	2.485	2.612	2.787	3.078	3.725
26	2.162	2.296	2.479	2.605	2.779	3.067	3.707
27	2.158	2.291	2.473	2.598	2.771	3.057	3.690
28	2.154	2.286	2.467	2.592	2.763	3.047	3.674
29	2.150	2.282	2.462	2.586	2.756	3.038	3.659
30	2.147	2.278	2.457	2.581	2.750	3.030	3.646
40	2.123	2.250	2.423	2.542	2.704	2.971	3.551
60	2.099	2.223	2.390	2.504	2.660	2.915	3.460
120	2.076	2.196	2.358	2.468	2.617	2.860	3.373
∞	2.054	2.170	2.326	2.432	2.576	2.807	3.291