UNIVERSITY OF SWAZILAND

SUPPLEMENTARY EXAMINATION PAPER 2014

TITLE OF PAPER	:	DISTRIBUTION THEORY
COURSE CODE	:	ST301
TIME ALLOWED	:	TWO (2) HOURS
REQUIREMENTS	:	CALCULATOR
INSTRUCTIONS	:	ANSWER ANY THREE QUESTIONS

Question 1 [20 marks, 3+3+6+2+6]

Assume $\{X_n\}_{n=0}^{\infty}$ is a Markov chain (MC) with transition probability matrix

$$\mathrm{P} = \begin{pmatrix} 0 & 0.3 & 0.2 & 0.5 \\ 0.3 & 0 & 0.5 & 0.2 \\ 0 & 0 & 0.4 & 0.6 \\ 0 & 0 & 0.3 & 0.7 \end{pmatrix}.$$

(a) Find the two step transition probability matrix.

(b) Suppose that the probability function of X_1 is given by the vector $\beta = (0, 0.5, 0, 0.5)$. Find the probability function of X_3.

(c) Classify the state space. For each class, determine whether it is recurrent or transient. Determine their periods.

(d) What does it mean by "irreducible"? Is this MC reducible?

(e) Find the long run proportions of times when the MC is in state 0, in state 2. (Do not blindly solve $\pi\mathrm{P} = \pi$).

Question 2 [20 marks, 6+4+10]

(a) There are 12 beads in an urn, where 5 of them are blue, 4 are red and 3 are yellow. Beads with the same colour are identical. You pick out 9 beads randomly from the urn without replacement.

(i) Find the probability that there are exactly 3 blue beads chosen.

(ii) Suppose now that you randomly pick out 9 beads from the urn, but with replacement. Find the probability of observing exactly two colours of beads, red and yellow, in the 9 chosen beads.

(b) Suppose that the genders of all children in a family are independent and that boys and girls are equally probable, that is, both have probability 0.5. Let the probability p_n that a family has exactly n children be $(1-p)p^n$ for $n = 0, 1, \cdots$ with p such that $0 < p < 1$. Show that the probability that a family contains exactly k boys, is given by

$$2(1-p)p^k(2-p)^{-(k+1)}.$$

Question 3 [20 marks,3+5+6+2+4]

(a) We define $N(t)$ to be the number of customers who have visited a shop since it opened at 9:00am. Our model is $N(t) \sim \text{Poisson}(\lambda t)$ for $t \geq 0$ when t is measured

(i) Give an expression for the probability that 20 customers have visited the shop by the time it closes at 5:00pm.

(ii) Let T be the time the first customer of the day visits the shop. By considering $P(T > t)$, show that T has an exponential distribution.

(iii) It is 10:00am and no customers have visited the shop so far today. Give an expression for the probability that the first customer of the day arrives between 10:00amand 11:00am. What do you notice about this expression?

(b) Consider a sample space $(\Omega, \mathfrak{F}, P)$ and events A, B and C. Using the axiom that the probability of a union of disjoint events is the sum of the probabilities of the individual events, and the definition of independence show that:

(i) if A and B are independent, then A and $\bar{B}$ are independent.

(ii) Define the conditional probability $P(B|C)$. Events A and B are said to be independent conditional on C if $P(A \cap B|C) = P(A|C)P(B|C)$. Show that if A, B and C are mutually independent, the A and B are independent conditional on C. Is conditional independence implied by the statement "A, B and C are pairwise independent"?

Question 4 [20 marks, 10+10]

(a) Let X and Y be independent, uniform random variables on [0, 1]. Find the density function and distribution function for $X + Y$.

(b) Suppose that random variable X has mgf M_X given by

$$M_X(t) = \frac{1}{8}e^t + \frac{2}{8}e^{2t} + \frac{5}{8}e^{3t}.$$

Find the probability distribution, and the expectation and variance of X.

Question 5 [20 marks, 3+4+3+4+6]

(a) Without a vaccine, the probability of contracting disease D_1 is 0.2, while for disease D_2 it is 0.05. An individual will not contract both diseases at the same time.

Vaccine A lowers the probability of contracting disease D_1 to 0.1, and disease D_2 to 0.02. Vaccine B lowers these probabilities to 0.05 and 0.04, respectively.

The proportion of the population who have not received either vaccine is 0.1, the proportion who have been vaccinated by vaccine A is 0.4, and the proportion vaccinated by vaccine B is 0.5.

(i) What is the probability that a particular patient has developed disease D_1?

(ii) Given a patient has developed disease D_1, what is the probability that he/she has been vaccinated with either vaccine A or B?

(b) Without a vaccine, the death rate is 0.1 for a patient with disease D_1, and 0.5 for a patient with disease D_2. If a patient receives vaccine A and still develops a disease, the death rate is 0.06 for disease D_1 and 0.1 for disease D_2. If a patient receives vaccine B and still develops a disease, the corresponding death rates are 0.01 and 0.2, respectively.

(i) Show that for any events A, B and C, we have

$$P(A \cap B \cap C) = P(C|A \cap B)P(B \cap A)P(A)$$

(ii) Find the probability that a particular individual satisfies the conditions that he/she is not vaccinated, develops disease D_2, and dies eventually.

(iii) Given that a patient developed disease D_2 and died, find the probability that the patient has not been vaccinated.