UNIVERSITY OF SWAZILAND

FINAL EXAMINATION PAPER, DECEMBER 2011

TITLE OF PAPER	:	DISTRIBUTION THEORY
COURSE CODE	:	ST301
TIME ALLOWED	:	TWO (2) HOURS
INSTRUCTIONS	:	ANSWER ANY THREE (3) QUESTIONS

Question 1 **[20 marks, 4+6+3+7]**

(a) The joint density of (X, Y) is

$$f_{X,Y}(x,y) = \begin{cases} x^k e^{-ky}, & 0 < x < y < \infty, \\ 0, & \text{otherwise.} \end{cases}$$

(i) Verify that this is a valid density when $k = 1$.

(ii) Find $f_X(x)$, $f_Y(y)$ and $f_{Y|X}(y|x)$, representing the marginal density of X, Y and the conditional density of Y given $X = x$ respectively.

(b) In a multiple-choice test, the probability that you know the answer to a question is 0.6. If you do not know the answer, you choose one at random. Suppose that there are 10 questions in the test and let C be the number of multiple choices per question (the test will have all the questions having the same number of choices C). Before taking the test, you do not know the value of C, but you know that it can be either 3 or 4, with probability 0.3 and 0.7 respectively.

(i) Find the probability that you know the answers to at least 80% of the questions.

(ii) Assume that $C = 3$. Find the conditional probability that you answer at least 90% of the questions correctly, given that you know the answers to at least 80% of the questions.

Question 2 **[20 marks, 3+3+4+2+4+4]**

(a) You have a coin and two dice. The coin is biased and comes up head with probability p and tail with probability $q = 1 - p$. Dice 1 (D_1) is a fair dice. Dice 2 (D_2) is a biased dice with

$$P(D_2 = 1) = P(D_2 = 6) = \frac{1}{12}$$
$$P(D_2 = 2) = P(D_2 = 5) = \frac{1}{6}$$
$$P(D_2 = 3) = P(D_2 = 4) = \frac{1}{4}$$

You flip the biased coin. If it comes up a head, you throw Dice 1. Otherwise you throw Dice 2. Let X denote the result of the dice throw.

(i) What is the probability that the outcome of the throw is 6?

(ii) Given the outcome is 6, what is the probability that the dice thrown was dice 1?

(iii) After the first throw of dice, you throw the dice you have just thrown again. What is the probability that the two throws add up to 3?

(b) Let Z be a random variable with density

$$f_Z(z) = \frac{1}{2}e^{-|z|}, \qquad \text{for } 0 < z < \infty.$$

(i) Show that f_Z is a valid density.

(ii) Find the moment generating function of Z and specify the interval where the MGF is well-defined.

(iii) By considering the cumulant generating function or otherwise, evaluate $E(Z)$ and $Var(Z)$.

Question 3 [20 marks, 4+6+4+6]

(a) Let $X \sim$ Bernuolli(p), that is, $P(X=x)=p^x(1-p)^{1-x}$, $x=0,1$. Find the moment generating function of X. Let $Y=NX$, when N Poisson(μ) with

$$P(N=n)=\frac{\mu^n e^{-\mu}}{n!} \quad n=0,1,\cdots,$$

and N is independent of X. Derive the moment generating function of Y.

(b) There are 12 beads in an urn, where 5 of them are blue, 4 are red and 3 are yellow. Beads with the same colour are identical. You pick out 9 beads randomly from the urn without replacement.

(i) Find the probability that there are exactly 3 blue beads chosen.

(ii) Suppose now that you randomly pick out 9 beads from the urn, but with replacement. Find the probability of observing exactly two colours of beads, red and yellow, in the 9 chosen beads.

Question 4 [20 marks, 3+4+3+4+6]

(a) Without a vaccine, the probability of contracting disease D_1 is 0.2, while for disease D_2 it is 0.05. An individual will not contract both diseases at the same time.

Vaccine A lowers the probability of contracting disease D_1 to 0.1, and disease D_2 to 0.02. Vaccine B lowers these probabilities to 0.05 and 0.04, respectively.

The proportion of the population who have not received either vaccine is 0.1, the proportion who have been vaccinated by vaccine A is 0.4, and the proportion vaccinated by vaccine B is 0.5.

(i) What is the probability that a particular patient has developed disease D_1?

(ii) Given a patient has developed disease D_1, what is the probability that he/she has been vaccinated with either vaccine A or B?

(b) Without a vaccine, the death rate is 0.1 for a patient with disease D_1, and 0.5 for a patient with disease D_2. If a patient receives vaccine A and still develops a disease, the death rate is 0.06 for disease D_1 and 0.1 for disease D_2. If a patient receives vaccine B and still develops a disease, the corresponding death rates are 0.01 and 0.2, respectively.

(i) Show that for any events A, B and C, we have

$$P(A\cap B\cap C)=P(C|A\cap B)P(B\cap A)P(A)$$

(ii) Find the probability that a particular individual satisfies the conditions that he/she is not vaccinated, develops disease D_2, and dies eventually.

(iii) Given that a patient developed disease D_2 and died, find the probability that the patient has not been vaccinated.

Question 5 [20 marks,10+4+6]

(a) Let $N \sim$ Poisson(μ). Define the random variable

$$Y = \begin{cases} X_1 \sim \text{Exponential}(\lambda) & N > 0; \\ X_2 \sim \text{Exponential}(2\lambda) & N = 0, \end{cases}$$

where N, X_1 and X_2 are independent of each other. Derive the moment generating function of Y, and find $E(Y)$ and Var(Y).

(b) Let X and Y be two independent standard normal random variables. Consider random variables U and V such that X and Y can be represented by

$$\begin{cases} X = U \cos V, \\ Y = U \sin V. \end{cases}$$

You are given some useful properties of sin and cos functions:

$$\frac{d}{dx}(\sin x) = \cos x, \quad \frac{d}{dx}(\cos x) = -\sin x, \quad \sin^2(x) + \cos^2(x) = 1.$$

Also, over $[0, 2\pi)$, $\sin \geq 0$ for $x \in [0, \pi]$ and $\cos x \geq 0$ for $xin[0, \pi/2]$ of $[3\pi/2, 2\pi)$.

(i) Give the respective ranges for U and V in order that the transformation defined is one to one. With this, find U and V in terms of X and Y.

(ii) Find the joint probability density of $f_{U,V}(u, v)$ of U and V.

Question 6 [20 marks,3+5+6+2+4]

(a) We define $N(t)$ to be the number of customers who have visited a shop since it opened at 9:00am. Our model is $N(t) \sim$ Poisson(λt) for $t \geq 0$ when t is measured

(i) Give an expression for the probability that 20 customers have visited the shop by the time it closes at 5:00pm.

(ii) Let T be the time the first customer of the day visits the shop. By considering $P(T > t)$, show that T has an exponential distribution.

(iii) It is 10:00am and no customers have visited the shop so far today. Give an expression for the probability that the first customer of the day arrives between 10:00amand 11:00am. What do you notice about this expression?

(b) Consider a sample space $(\Omega, \mathfrak{F}, P)$ and events A, B and C. Using the axiom that the probability of a union of disjoint events is the sum of the probabilities of the individual events, and the definition of independence show that:

(i) if A and B are independent, then A and $\bar{B}$ are independent.

(ii) Define the conditional probability $P(B|C)$. Events A and B are said to be independent conditional on C if $P(A \cap B|C) = P(A|C)P(B|C)$. Show that if A, B and C are mutually independent, the A and B are independent conditional on C. Is conditional independence implied by the statement "A, B and C are pairwise independent"?