UNIVERSITY OF SWAZILAND

SUPPLEMENTARY EXAMINATION PAPER 2011

TITLE OF PAPER	:	DISTRIBUTION THEORY
COURSE CODE	:	ST301
TIME ALLOWED	:	TWO (2) HOURS
REQUIREMENTS	:	CALCULATOR AND STATISTICAL TABLES
INSTRUCTIONS	:	ANSWER ANY THREE QUESTIONS

Question 1 [20 marks, 6+6+8]

(a) For each of the joint pdf's below, determine the conditional density function of Y given $X = x$ for all x such that the function is defined.

(i) $f(x,y) = \begin{cases} \lambda^2 e^{-\lambda y} & 0 \le x \le y \\ 0 & \text{elsewhere} \end{cases}$

(ii) $f(x,y) = \begin{cases} xe^{-x(y+1)} & x, y \ge 0 \\ 0 & \text{elsewhere} \end{cases}$

(b) Breakdowns in a factory occur with Poisson rate λ. The time T to effect a repair (assume T is independent of the breakdown process) has the distribution

$$\mathbb{P}(T = j) = q^{j-1}p \qquad j = 1, 2, \cdots \qquad 0 < p < 1,\ q = 1 - p$$

Calculate the expected number of breakdowns while one repair is being effected.

Question 2 [20 marks, 6+8+6]

(a) Consider the following suggested model for the lifetime of an electrical component. Each component has a quality factor q; and the distribution of the lifetime of the component has pdf

$$f(x) = \begin{cases} qe^{-qx} & x > 0 \\ 0 & \text{elsewhere} \end{cases}$$

where $q > 0$. The value of q varies between components. The quality factor of a randomly chosen component has pdf

$$f(q) = \begin{cases} \beta e^{-\beta q} & q > 0 \\ 0 & \text{elsewhere} \end{cases}$$

where $\beta > 0$ is a parameter. Find the pdf of the lifetime of a randomly chosen component.

(b) X has pdf

$$f(x) = \begin{cases} \frac{kx^3}{(1+2x)^6} & x > 0 \\ 0 & \text{elsewhere} \end{cases}$$

Find the pdf of $Y = \frac{2X}{1+2X}$ and identify the constant k.

(c) X_1, X_2 and X_3 are independent, each uniformly distributed on the interval $(0, 1)$. Find $\mathbb{P}(X_1 < X_2 < X_3)$.

Question 3 [20 marks, 12+8]

(a) The count random variables X and Y are independent and Poisson distributed with parameters λ and μ respectively, i.e.

$$P(X = k) = \frac{\lambda^k e^{-\lambda}}{k!}, \qquad P(Y = k) = \frac{\mu^k e^{-\mu}}{k!}, \qquad k = 0, 1, 2, \cdots, \infty.$$

Show that $Z = X + Y$ is Poisson distributed with parameter $(\lambda + \mu)$. Show also that the conditional distribution of X, given that $X + Y = n$, is binomial, and determine the parameters.

(b) It is said that a random variable X has a *Pareto distribution with parameters* x_0 *and* α ($x_0 > 0$ and $\alpha > 0$) if X has a continuous distribution for which the p.d.f is

$$f_X(x) = \begin{cases} \frac{\alpha x_0^\alpha}{x^{\alpha+1}}, & x \geq x_0, \\ 0, & x \leq x_0. \end{cases}$$

Show that if X has this Pareto distribution, then the random variable $\log(X/x_0)$ has an exponential distribution with parameter α.

Question 4 **[20 marks, 6+5+5+4]**

(a) The count random variable X has probability generating function (PGF)

$$G_X(s) = \frac{1 - s^{M+1}}{(M+1)(1-s)}$$

where M is a positive integer. Find the probability function of X.

(b) Consider random variables X and Y with joint density function

$$f_{X,Y}(x,y) = \begin{cases} k(3x-2) & \text{for } 0 < y < x < 2, \\ 0 & \text{otherwise.} \end{cases}$$

(i) Find k.

(ii) Find $f_X(x)$. Hence evaluate $E(X)$.

(iii) Evaluate $P(2Y > X)$.

Question 5 **[20 marks, 8+4+8]**

(a) A measurement, X, has probability density function given by

$$f(x) = \frac{\lambda x^{\lambda-1}}{\theta^\lambda} \exp(-(x/\theta)^\lambda) \qquad x > 0$$

where λ and θ are positive parameters. Show that the $100p$ percentile of this distribution is $\theta(-\log(1-p))^{1/\lambda}$. Hence use the respective estimates 4.425 and 5.575 of the median and upper quartile to deduce that the values of λ and θ can be estimated as approximately 3 and 5 respectively.

(b) In an accelerated life experiment, the times to failure, in hours, of a certain type of device have probability density function

$$f(x) = \nu^2 x e^{-\nu x}$$

for $x > 0$. Show that the mean time to failure is $2/\nu$.

(c) The continuous random variables X and Y have joint probability density function $f(x,y) = kxy$ if $0 < x < y < 1$, with $f(x,y) = 0$ elsewhere, where k is a constant. Evaluate k, and find the marginal probability densities of X and Y. Say, with a reason, whether or not X and Y are independent.

Appendix

- A continuous non-negative random variable X is distributed Gamma(α,λ), with p.d.f.

$$f(x) = \frac{\lambda^{\alpha} x^{\alpha-1} \exp(-\lambda x)}{\Gamma(\alpha)}, \qquad x \geq 0; \quad \alpha, \lambda > 0$$

- The 2-parameter Weibull distribution has the c.d.f.

$$F(x) = 1 - \exp\left\{-\left(\frac{x}{b}\right)^{c}\right\}, \qquad x \geq 0.$$

- The function

$$\Gamma(p) = \int_0^{\infty} t^{p-1} e^{-t} dt, \quad p > 0,$$

has the properties

$$\Gamma(p+1) = p\Gamma(p) : \quad \Gamma(1/2) = \sqrt{\pi} : \quad \Gamma(n+1) = n!, \qquad \text{n integer } \geq 0.$$