UNIVERSITY OF SWAZILAND

FINAL EXAMINATION PAPER 2010

TITLE OF PAPER	:	DISTRIBUTION THEORY
COURSE CODE	:	ST301
TIME ALLOWED	:	TWO (2) HOURS
REQUIREMENTS	:	CALCULATOR AND STATISTICAL TABLES
INSTRUCTIONS	:	ANSWER ANY THREE QUESTIONS

Question 1 [20 marks, 12+8]

(a) The count random variables X and Y are independent and Poisson distributed with parameters λ and μ respectively, i.e.

$$P(X=k)=\frac{\lambda^k e^{-\lambda}}{k!}, \qquad P(Y=k)=\frac{\mu^k e^{-\mu}}{k!}, \qquad k=0,1,2,\cdots,\infty.$$

Show that $Z=X+Y$ is Poisson distributed with parameter $(\lambda+\mu)$. Show also that the conditional distribution of X, given that $X+Y=n$, is binomial, and determine the parameters.

(b) It is said that a random variable X has a *Pareto distribution with parameters x_0 and α* ($x_0>0$ and $\alpha>0$) if X has a continuous distribution for which the p.d.f is

$$f_X(x)=\begin{cases}\frac{\alpha x_0^\alpha}{x^{\alpha+1}}, & x\geq x_0,\\ 0, & x\leq x_0.\end{cases}$$

Show that if X has this Pareto distribution, then the random variable $\log(X/x_0)$ has an exponential distribution with parameter α.

Question 2 [20 marks, 6+6+8]

(a) A prisoner is trapped in a dark cell containing 3 doors. The first door leads to a tunnel that returns him to his cell after 2 days of travel, the second leads to a tunnel that returns him to his cell after 4 days, while the third door leads to freedom after 1 day. If it is assumed that the prisoner will always select doors 1,2 and 3 with probabilities 0.5, 0.3 and 0.2 respectively, what is the expected time for the prisoner to reach freedom?

(b) Suppose the number of car accidents in a year is Poisson distributed with parameter λ and the probabilities of an accident involving 1, 2, 3, 4 cars are 0.38, 0.55, 0.05, 0.02 respectively. Obtain an expression for the probability generating function (PGF) of X, the total number of cars involved in accidents during a year, and derive from it the expected value of X.

(c) Consider a population of individuals which can die or reproduce independently of each other with fixed generation time. Suppose the population is of size 1 initially. Let the random variable C denote the number of children of one individual where

$$\mathbf{P}(C=k)=\left(\frac{1}{2}\right)^{k-1}, \qquad k=0,1,2,\cdots$$

with PGF $G(s)$. Let the random variable X_n be the size of the n^{th} generation with PGF $G_n(s)$. Find the PGFs $G_0(s)$, $G_1(s)$, $G_2(s)$, and $G_3(s)$.

Question 3 [20 marks, 14+2+4]

(a) The lifetime X of a certain device has c.d.f.

$$F(x)=1-e^{-\lambda x^2}, \qquad x\geq 0,\ \lambda>0.$$

Derive the p.d.f. of X, $f(x)$, and determine its mean, variance and mode. Also determine the hazard rate function $r(x)$, and briefly explain its significance.

Note: the function

$$\Gamma(p) = \int_0^\infty t^{p-1}e^{-t}dt, \quad p > 0,$$

has the properties

$$\Gamma(p+1) = p\Gamma(p) : \quad \Gamma(1/2) = \sqrt{\pi} : \quad \Gamma(n+1) = n!, \quad \text{n integer} \geq 0.$$

(b) Show that

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k} = 0$$

(c) Verify the following identity

$$\sum_{k=1}^{n}\binom{n}{k}k = 2^{n-1}n$$

Question 4 [20 marks, 4+4+4+4+4]

(a) Let X be a random variable with probability density function

$$f_X(x;\mu) = \frac{1}{\mu}\exp(-x/\mu),$$

for $x > 0$ and zero elsewhere.

(i) Calculate the mean and variance of X.

(ii) Calculate the mean and variance of X conditional on $X < 8$.

(b) Suppose the continuous random variables (X, Y, Z) have joint

$$f(x,y,z) = Kxyz^2, \quad 0 \leq x, y \leq 1, \quad 0 \leq z \leq 3.$$

(i) Show that the constant $K = \frac{4}{9}$.

(ii) Find the marginal probability density function of Y and hence show that $E(Y) = \frac{2}{3}$.

(iii) Show that the marginal joint probability density function of (X, Z) is

$$f_{X,Z}(x,z) = \frac{2}{9}xz^2 \quad 0 \leq x \leq 1, \quad 0 \leq z \leq 3.$$

Question 5 [20 marks,7+13]

(a) An insurance company offers annual motor-car insurance based on a "no claims discount" system with levels of discount 0%, 30% and 60%. A policyholder who makes no claims during the year either moves to the next higher level of discount or remains at the top level. If there is exactly one claim during the year, the policyholder either moves down one level or stays at the bottom level (0%). If there is more than one claim during the year, the policyholder either moves down to or stays at the bottom level. For a particular policyholder, it may be assumed that claims arise in a Poisson process at rate $\lambda > 0$. Explain why the situation described above is suitable for modelling in terms of a Markov chain with three states, and write down the transition probability matrix in terms of λ.

(b) The random variables X and Y have the (trinomial) probability mass function

$$p_{X,Y}(x,y) = \frac{n!}{x!y!(n-x-y)!} p^x q^y (1-p-q)^{n-x-y},$$

for $x, y = 0, 1, \cdots, n$ subject to $x + y \leq n$.

(i) Show that the marginal probability mass function of X is of binomial form, and identify its parameters.

(ii) Suppose x $(0 \leq x < n)$ is given. Show that the conditional distribution of Y, given $X = x$, is of binomial form, and write down $E(Y|X = x)$.

(iii) Obtain $E(XY)$ and deduce that $\text{Cov}(X, Y) = -npq$.

Question 6 **[20 marks,8+6+6]**

(a) Let Y be distributed uniformly on $[a, b]$ that is

$$f_Y(y) = \begin{cases} 1/(b-a), & a \leq y \leq b \\ 0, & \text{otherwise.} \end{cases}$$

Find the MGF of Y and hence calculate $E(Y)$ and $\text{Var}(Y)$.

(b) The count random variable X has probability generating function (PGF)

$$G_X(s) = \frac{1 - s^{M+1}}{(M+1)(1-s)}$$

where M is a positive integer. Find the probability function of X.

(c) A homogeneous Markov chain $\{X_n : n = 0, 1, \cdots\}$ has states $\{0, 1, 2\}$ and transition probability matrix

$$\mathbf{P} = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{pmatrix}$$

At time $n = 0$, the system is equally likely to be in any of the states $0, 1, 2$. Find $P(X_2 = 1)$ and $P(X_2 = 2)$.