UNIVERSITY OF SWAZILAND

FACULTY OF SCIENCE AND ENGINEERING

DEPARTMENT OF PHYSICS

MAIN EXAMINATION 2021

TITLE OF PAPER: ADVANCED STATISTICAL PHYSICS

COURSE NUMBER: PHY604

TIME ALLOWED: THREE HOURS

ANSWER ALL QUESTIONS

NB: THE QUESTIONS **DO NOT** HAVE THE SAME MARKS

THE APPENDIX CONTAINS USEFUL FORMULAE

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QUESTION ONE [40 Marks]

Consider a system consisting of N independent parts. An extensive variable Y has an average vale $\langle Y \rangle$ and moments $\langle (\Delta Y)^m \rangle$, where $\Delta Y = Y - \langle Y \rangle$.

(a) Determine the exponent a for the odd moments:

$$\frac{\langle (\Delta Y)^{2m+1} \rangle^{1/(2m+1)}}{\langle Y \rangle} \sim \frac{1}{N^a}$$

and show that $a > b$ where

$$\frac{\langle (\Delta Y)^{2m} \rangle^{1/(2m)}}{\langle Y \rangle} \sim \frac{1}{N^b}$$

[20 marks]

(b) Construct the probability weight using all the integer moments, odd and even. Show that the odd moments do not contribute to the distribution function as $N \to \infty$.

A trick that may be helpful (and which is used in the standard derivation of the central-limit theorem) is to consider the variable $X \equiv Y/\sqrt{N}$.

[20 marks]

QUESTION TWO [30 Marks]

Consider the canonical ensemble for a monocomponent simple fluid, the probability of a fluctuation away from thermodynamic equilibrium is given by:

$$P \propto e^{-(\Delta T \Delta S - \Delta p \Delta V)/2k_B T}$$

(a) From this expression, find $P(\Delta T, \Delta p)$

[10 marks]

(b) Show that one can also write the thermodynamic potential in (a) in terms of the Gibbs free energy G is given by:

$$P(\Delta T, \Delta p) \sim e^{\Delta G/k_B T}$$

[10 marks]

(c) Show that the energy fluctuations in the canonical ensemble depend on the heat capacity C_V, i.e.

$$(\Delta E)^2 = k_B T^2 C_V$$

[10 marks]

QUESTION THREE [30 Marks]

A rare analytic solution to an interacting system involves the infinite-range Ising model of a ferromagnet. Consider spins $S_i = \pm 1$ that are located on $i = 1, 2, ...N$ sites of a regular lattice. All spins interact equally with each other, through a coupling positive constant J_∞. As well, there is a constant external magnetic field H which tends to align the spins in the direction (sign) of the field. The energy of interaction for the system is given by:

$$E = -\frac{J_\infty}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} S_i S_j - H\sum_{i=1}^{N} S_i$$

and the partition function is

$$Z = \sum_{S_1=-1}^{+1}\sum_{S_2=-1}^{+1}\cdots\sum_{S_N=-1}^{+1} e^{-E/k_B T}$$

where k_B is the Boltzmann constant and T is the temperature.

(a) The fact that all spins interact with each other with equal strength (without regard to how far apart the spins may be) is artificial. For instance, why does the model only make sense if $J_\infty = J/N$, where J is a constant independent of N?

[5 marks]

(b) Show that the partition function for the system above is

$$Z = \int_{-\infty}^{\infty} \frac{dy}{\sqrt{2\pi k_B T/NJ}} e^{-NL(y)/k_B T}$$

(as a hint, let $x = \sum_{i=1}^{N} S_i$), and

$$L = \frac{J}{2}y^2 - k_B T \ln 2\, cosh(H + Jy)/k_B T$$

The sums in the partition function have been replaced by an integral over the auxiliary variable y.

[10 marks]

(c) In the thermodynamic limit, $N \to \infty$, show that the partition function can be evaluated exactly as

$$Z = \sum_{\infty} \exp(-NL(N, J, T, y_\alpha))/k_B T$$

find the equation satistied by y, and explain what the index α is. What is the probability of being in the state specified by y_α?

[10 marks]

(d) How can you show that there is a phase transition?

[4 marks]

(e) Comment about the nature of this model

[1 mark]

APPENDIX

Useful formulae

The partition function and various probability weights:

$$Z = \sum_{states} e^{-E_{state}/k_B T} = e^{-F/k_B T}$$

For a normalized Gaussian, $\int_{-\infty}^{\infty} dx p(x)$, where the weight is:

$$p(x) = \frac{1}{\sqrt{2\pi\sigma}} e^{-x^2/(2\sigma^2)}$$

where $\langle x \rangle = 0$ and $\langle x^2 \rangle = \sigma^2$. Also,

$$p \propto e^{S/k_B}, \quad p \propto e^{F/k_B T}, \quad p = e^{-E_{state}/k_B T}/Z$$

Thermodynamic relations:

$$TdS = dE + PdV, \quad F = E - TS, \quad \kappa_T = -\frac{1}{V}\left(\frac{\partial V}{\partial P}\right),$$

$$P = -\left(\frac{\partial F}{\partial T}\right), \quad C_v = -T\left(\frac{\partial^2 F}{\partial T^2}\right), \quad \left(\frac{\partial T}{\partial V}\right)_S = -\left(\frac{\partial P}{\partial S}\right)_V,$$

$$\left(\frac{\partial S}{\partial V}\right)_T = \left(\frac{\partial P}{\partial T}\right)_V, \quad \sigma = \left(\frac{\partial F}{\partial A}\right)_{T,V}$$

$$dU = TdS - pdV + \mu dN \quad dG = SdT + Vdp + dN$$

$$-\left(\frac{\partial S}{\partial P}\right)_T = \left(\frac{\partial V}{\partial T}\right)_P, \quad \left(\frac{\partial T}{\partial V}\right)_S = -\left(\frac{\partial P}{\partial S}\right)_V, \quad \left(\frac{\partial T}{\partial P}\right)_S = \left(\frac{\partial V}{\partial S}\right)_p,$$

and heat capacities, compressibilities, and coefficient of thermal expansion

$$C_V = T\left(\frac{\partial S}{\partial T}\right)_V, \quad C_p = T\left(\frac{\partial S}{\partial T}\right)_p, \quad \kappa_T = -\frac{1}{V}\left(\frac{\partial V}{\partial p}\right)_T, \quad \alpha = -\frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_p,$$

Also note the thermodynamic sum rule:

$$\int d\vec{r} C(r) = n^2 k_B T \kappa_T$$

Thermodynamic fluctuations:

$$\langle(\Delta T)^2\rangle = k_B T^2/C_V, \quad \langle(\Delta T)(\Delta V)\rangle = 0, \quad \langle(\Delta V)^2\rangle = k_B T V \kappa_T$$

Fourier relations (in d dimensions, change to $(d-1)$ for surface fluctuations, for example). Note all integrals are bounded on small length scales by an ultraviolet cutoff, $|\vec{k}| < \Lambda$, where $2\pi/\Lambda \sim$ (a few nanometers), and if necessary on large length scales by an infrared cutoff $|\vec{k}| > (2\pi/L)$.

$$\psi(\vec{x}) = \int \frac{d^d k}{(2\pi)^d} e^{i\vec{k}.\vec{x}} \hat{\psi}(\vec{k}) = \frac{1}{L^d} \sum_{\vec{k}} e^{i\vec{k}.\vec{x}} \hat{\psi}(\vec{k}), \quad \hat{\psi}(\vec{k}) = \int d^d x e^{-i\vec{k}.\vec{x}} \psi(\vec{x})$$

Closure requires the Dirac delta functions:

$$\delta(\vec{x}) = \int \frac{d^d k}{(2\pi)^d} e^{i\vec{k}.\vec{x}} = \frac{1}{L^d} \sum_{\vec{k}} e^{i\vec{k}.\vec{x}}, \quad \delta(\vec{k}) = \int \frac{d^d x}{(2\pi)^d} e^{i\vec{k}.\vec{x}}$$

The Knoecker delta satisfies:

$$\delta_{\vec{k},0} = \frac{1}{L^d} \int d^d x e^{i\vec{k}.\vec{x}},$$

which is 1 if $\vec{k} = 0$, and zero otherwise. These relations all use the density of states in k space as $(L/2\pi)^d$. So, the discrete to continuum limit is

$$\sum_{\vec{k}} \to \left(\frac{L}{2\pi}\right)^d \int d^d k, \quad \delta_{\vec{k},0} = \left(\frac{2\pi}{L}\right)^d \delta(\vec{k})$$

and vice versa.

Handy integrals and sums

$$\int_{-\infty}^{\infty} dx e^{-x^2}\, dx = \sqrt{\pi}, \quad \sum_{n=0}^{\infty} \frac{y^n}{n!} = e^y, \quad \sum_{n=0}^{\infty} \epsilon^n = 1/(1-\epsilon)$$

where $|\epsilon| < 1$.

$$\exp\left(\frac{a}{2N} x^2\right) = \int_{-\infty}^{\infty} \frac{dy}{\sqrt{2\pi/Na}} e^{-\frac{Na}{2} y^2 + axy} \qquad \text{Re } a > 0$$