UNIVERSITY OF ESWATINI

FACULTY OF SCIENCE AND ENGINEERING

DEPARTMENT OF PHYSICS

MAIN EXAMINATION 2020/2021

TITLE OF PAPER: ADVANCED QUANTUM MECHANICS

COURSE NUMBER: PHY493

TIME ALLOWED: THREE HOURS

INSTRUCTIONS: ANSWER ALL QUESTIONS IN SECTION A AND ANY FOUR OUT OF FIVE QUESTIONS IN SECTION B.

MARKS FOR EACH QUESTION ARE IN THE RIGHT HAND MARGIN.

THIS PAPER HAS 8 PAGES INCLUDING THE COVER PAGE AND FORMULA SHEET.

DO NOT OPEN THE PAPER UNTIL PERMISSION HAS BEEN GIVEN BY THE CHIEF INVIGILATOR.

Section A

ANSWER ALL QUESTIONS IN THIS SECTION.

A1. Consider an operator

$$A = |a_1\rangle\langle a_1| + |a_2\rangle\langle a_2| + |a_3\rangle\langle a_3| - i\,|a_1\rangle\langle a_2| - |a_1\rangle\langle a_3| + i\,|a_2\rangle\langle a_1| - |a_3\rangle\langle a_1|$$

where $|a_1\rangle, |a_2\rangle$ and $|a_3\rangle$ form a complete and orthonormal basis. Is A Hermitian?

(3 marks)

A2. Find the 3 x 3 matrix representation A (in question A1.) in the $|a_1\rangle, |a_2\rangle\,|a_3\rangle$ basis.

(3 marks)

A3. Find the eigenvalues of the matrix in A2.

(3 marks)

A4. *Let $\psi_n(x)$ denote the orthonormal stationary states of a system corresponding to the energy E_n. Suppose that the normalized wave function of the system at time $t = 0$ is $\psi(x,0)$ and suppose that a measurement of the energy yields the value E_1 with probability $\frac{1}{2}$, E_2 with probability $\frac{3}{8}$, and E_3 with probability $\frac{1}{8}$. Write the most general expansion for $\psi(x,0)$ consistent with this information.*

$$\Psi(x,0) = \frac{e^{i\theta_1}}{\sqrt{2}}\Psi_1(x) + \frac{\sqrt{3}e^{i\theta_2}}{\sqrt{8}}\Psi_2(x) + \frac{e^{i\theta_3}}{\sqrt{8}}\Psi_3(x)$$

where θ_1, θ_2 and θ_3 are real numbers.

(5 marks)

A5. What is an antiunitary transformation?

(3 marks)

A6. Is $\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$ a pure or mixed state?

(3 marks)

Total = 20 marks

Section B

ANSWER ANY FOUR OUT OF FIVE QUESTIONS IN THIS SECTION.

QUESTION 1

A simple harmonic oscillator in one dimension is defined by the Hamiltonian

$$H = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right)$$

where the raising and lowering operators are given respectively by

$$a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}x - \frac{i}{\sqrt{2m\omega\hbar}}p$$
$$a = \sqrt{\frac{m\omega}{2\hbar}}x + \frac{i}{\sqrt{2m\omega\hbar}}p$$

with $[x,p] = i\hbar$ and $[a,a^\dagger] = 1$. Let $|n\rangle$ be a normalized (i.e $\langle n|n\rangle = 1$) eigenvector of the Hamiltonian with an energy eigenvalue of E_n (i.e $H|n\rangle = E_n|n\rangle$).

(a) Show that $[H,a] = -\hbar\omega a$ and $[H,a^\dagger] = \hbar\omega a^\dagger$.

(4 marks)

(b) Using the commutation relation in (a), show that $a|n\rangle$ is an eigenvector of H with eigenvalue $E_n - \hbar\omega$, and $a^\dagger|n\rangle$ is an eigenvector of H with eigenvalue $E_n + \hbar\omega$.

(4 marks)

(c) Using the relations

$$a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle \text{ and } a|n\rangle = \sqrt{n}|n-1\rangle\,,$$

calculate explicitly the expectation values $\langle p\rangle\,, \langle p^2\rangle\,, \langle x\rangle$ and $\langle x^2\rangle$ for a simple harmonic in the state $|n\rangle$. Obtain an expression for the product $\Delta x\Delta p$ as a function of n (where $\Delta x = \sqrt{\langle x^2\rangle - \langle x^2\rangle}$), and confirm that it complies with Heisenberg's uncertainty relation.

(8 marks)

(d) Now consider the eigenvectors $|n\rangle$ and $|m\rangle$ where $n \neq m$. Find the values of n and m for which the inner product $\langle n|x^2|m\rangle$ vanishes and the values for which this inner product does not vanish. (Show your working)

(4 marks)

(Total = 20 marks)

QUESTION 2

Consider the following relations for angular momentum operators:

$$L_+ = L_x + iL_y \qquad L_- = L_x - iL_y$$

$$L_+ = \hbar\sqrt{(\ell - m)(\ell + m + 1)}\,|\ell, m+1\rangle \quad L_- = \hbar\sqrt{(\ell + m)(\ell - m + 1)}\,|\ell, m-1\rangle$$

$$L_z\,|\ell m\rangle = m\hbar|\ell m\rangle \qquad L^2\,|\ell m\rangle = \ell(\ell+1)\hbar^2|\ell m\rangle$$

For a system with orbital angular momentum $\ell = 1$ the eigenvectors $|\ell m\rangle$ can be labelled by the eigenvalue m alone, and so one has three eigenvectors $|1\rangle_z$, $|0\rangle_z$ and $|-1\rangle_z$ for $m = 1, 0, -1$ respectively.

(a) Using the above relations for angular momentum operators, show that the matrix representation of L_x for $\ell = 1$ in the basis of eigenvectors of L_z and L^2 is given by

$$L_x = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

(7 marks)

(b) Now find the eigenvalues and (normalized) column vector representations of the eigenvectors of L_x, neglecting global phase factors.

(10 marks)

(c) Calculate the probability that a measurement of L_x will give zero for a system that is in the state

$$|\Psi\rangle = \frac{1}{\sqrt{14}}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

(3 marks)

(Total = 20 marks)

QUESTION 3

Consider an isotropic harmonic oscillator in two dimensions. The Hamiltonian is give by

$$H_0 = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{m\omega^2}{2}(x^2 + y^2)$$

(a) What are the energies of the three lowest-lying states? Is there any degeneracy?

(4 marks)

(b) We now apply a perturbation

$$V = \delta m\omega^2 xy,$$

where δ is a dimensionless real number much smaller than unity. Find the zeroth-order energy eigenket and the corresponding energy to first order [that is, the unperturbed energy obtained in (a) plus the first-order energy shift] for each of the three lowest-lying states.

(8 marks)

(c) Solve $H_0 + V$ problem exactly. Compare with the perturbation result obtained in (b). [You may use $\langle n'|x|n\rangle = \sqrt{\hbar/2m\omega}(\sqrt{n+1}\delta_{n',n+1} + \sqrt{n}\delta_{n',n-1}]$

(8 marks)

QUESTION 4

(a) Calculate the differential cross section in the Born approximation for the potential $V(r) = \frac{1}{r}V_0 e^{-r/R}$, known as the Yukawa potential.

(8 marks)

(b) Calculate the total cross section.

(5 marks)

(c) Find the relation between V_0 and R so that the Born approximation is valid.

(7 marks)

QUESTION 5

Consider the field theory of a complex-valued scalar field obeying the Klein-Gordon equation. The action of this theory is

$$S = \int d^4x \left(\partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi\right)$$

(a) Find the equation of motion.

(5 marks)

(b) The theory contains two sets of particles. Rewrite the conserved charge

$$Q = \int d^3x \frac{i}{2}(\phi^* \pi^* - \pi\phi)$$

in terms of creation and annihilation operators, and evaluate the charge of the particle of each type.

(15 marks)

(Total = 20 marks)