UNIVERSITY OF ESWATINI

FACULTY OF SCIENCE AND ENGINEERING

DEPARTMENT OF PHYSICS

MAIN EXAMINATION 2019/2020

TITLE OF PAPER: ADVANCED QUANTUM MECHANICS

COURSE NUMBER: PHY607

TIME ALLOWED: THREE HOURS

INSTRUCTIONS: ANSWER ALL QUESTIONS IN SECTION A AND ANY FOUR OUT OF FIVE QUESTIONS IN SECTION B.

MARKS FOR EACH QUESTION ARE IN THE RIGHT HAND MARGIN.

THIS PAPER HAS 9 PAGES INCLUDING THE COVER PAGE AND FORMULA SHEET.

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Section A

ANSWER ALL QUESTIONS IN THIS SECTION.

A1. Write down the adjoint of the following expression involving bras and kets, in which a_i are complex scalars, and $\hat{\Omega}$ and $\hat{\Lambda}$ are operators:

$$a_1^* |v\rangle + a_2 |w\rangle \langle p|q\rangle + a_3\hat{\Omega}\hat{\Lambda} |r\rangle + a_4\hat{\Lambda} |u\rangle$$

(3 marks)

A2. Consider an operator

$$\hat{A} = |\phi_1><\phi_1| + |\phi_2><\phi_2| + |\phi_3><\phi_3| - i|\phi_1><\phi_2| - |\phi_1><\phi_3| + i|\phi_2><\phi_1| - |\phi_3><\phi_1|$$

where $|\phi_1>, |\phi_2>$ and $|\phi_3>$ form a complete and orthonormal basis. Find the 3 x 3 matrix representation $\hat{A}$ in the $|\phi_1>, |\phi_2>, |\phi_3>$ basis.

(3 marks)

A3. Find the eigenvalues of the matrix in A2.

(3 marks)

A4. Is $\hat{A}$ in A2. Hermitian?

(2 marks)

A5. Complete the table below to show the difference between Schrodinger Picture and Heisenberg Picture:

	Schrodinger Picture (moving/stationary)	Heisenberg Picture (moving/stationary)
State ket		
Observable		
Base ket		

(3 marks)

A6. Is $\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$ a pure or mixed state?

(3 marks)

A7. State Noether's theorem.

(3 marks)

Total = 20 marks

Section B

ANSWER ANY FOUR OUT OF FIVE QUESTIONS IN THIS SECTION.

QUESTION 1

A simple harmonic oscillator in one dimension is defined by the Hamiltonian

$$\hat{H} = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right)$$

where the raising and lowering operators are given respectively by

$$a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\hat{x} - \frac{i}{\sqrt{2m\omega\hbar}}\hat{p}$$
$$a = \sqrt{\frac{m\omega}{2\hbar}}\hat{x} + \frac{i}{\sqrt{2m\omega\hbar}}\hat{p}$$

with $[\hat{x}, \hat{p}] = i\hbar$ and $[a, a^\dagger] = 1$. Let $|n\rangle$ be normalized (i.e $\langle n|n\rangle = 1$) eigenvector of the Hamiltonian with an energy eigenvalue of E_n (i.e $\hat{H}|n\rangle = E_n|n\rangle$).

(a) Show that $[\hat{H}, a] = -\hbar\omega a$ and $[\hat{H}, a^\dagger] = \hbar\omega a^\dagger$.

(4 marks)

(b) Using the commutation relation in (a), show that $a|n\rangle$ is an eigenvector of $\hat{H}$ with eigenvalue $E_n - \hbar\omega$, and $a^\dagger|n\rangle$ is an eigenvector of $\hat{H}$ with eigenvalue $E_n + \hbar\omega$.

(4 marks)

(c) Using the relations

$$a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle \quad \text{and} \quad a|n\rangle = \sqrt{n}|n-1\rangle,$$

calculate explicitly the expectation values $\langle\hat{p}\rangle, \langle\hat{p}^2\rangle, \langle\hat{x}\rangle$ and $\langle\hat{x}^2\rangle$ for a simple harmonic in the state $|n\rangle$. Obtain an expression for the product $\Delta x \Delta p$ as a function of n (where $\Delta x = \sqrt{\langle\hat{x}^2\rangle - \langle\hat{x}^2\rangle}$), and confirm that it complies with Heisenberg's uncertainty relation.

(8 marks)

(d) Now consider the eigenvector $|n\rangle$ and $|m\rangle$ where $n \neq m$. Find the values of n and m for which the inner product $\langle n|\hat{x}^2|m\rangle$ vanishes and the values for which this inner product does not vanish. (Show your working)

(4 marks)

(Total = 20 marks)

QUESTION 2

(a) Write down an expression for the most general state ($|\Psi\rangle_{AB}$) in a Hilbert space $V_A \otimes V_B$, where V_A and V_B are Hilbert spaces with orthonormal bases $\{|i\rangle_A\}$ and $\{|j\rangle_B\}$ respectively. Then give the condition for this general state to be separable and the condition for this state to be entangled.

(6 marks)

(b) Suppose that Alice has a single qubit sate $|\phi\rangle = \alpha|0\rangle + \beta|1\rangle$ which she wishes to teleport to her friend Bob. To do this she creates an entangled state

$$|\Psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle$$

keeping the first (left) qubit and sending the second (right) qubit to Bob. Show that twice the product state $|\phi\rangle\,|\Psi\rangle$ can be written as

$$\begin{aligned} 2|\phi\rangle\,|\Phi\rangle &= |B_0\rangle\,(\alpha\,|0\rangle + \beta|1\rangle) + |B_1\rangle\,(\alpha\,|1\rangle + \beta|0\rangle) \\ &\quad + |B_2\rangle\,(\alpha\,|0\rangle - \beta|1\rangle) + |B_3\rangle\,(\alpha\,|1\rangle - \beta|0\rangle) \end{aligned}$$

where

$$|B_0\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle), \qquad |B_1\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle),$$

$$|B_2\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle), \qquad |B_3\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle).$$

(8 marks)

(c) Now show that the operators

$$\hat{I} = |0\rangle\,\langle 0| + |1\rangle\,\langle 1|\,, \qquad \hat{X} = |0\rangle\,\langle 1| + |1\rangle\,\langle 0|$$

$$\hat{Y} = |0\rangle\,\langle 1| - |1\rangle\,\langle 0|\,, \qquad \hat{Z} = |0\rangle\,\langle 0| - |1\rangle\,\langle 0|$$

can be used to transform the single qubit parts of the product state above into $|\phi\rangle$. That is, show that

$$\hat{I}(\alpha|0\rangle + \beta|1\rangle) = |\phi\rangle\,, \qquad \hat{X}(\alpha|1\rangle + \beta|0\rangle) = |\phi\rangle\,,$$

$$\hat{Y}(\alpha|1\rangle - \beta|0\rangle) = |\phi\rangle\,, \qquad \hat{Z}(\alpha|0\rangle - \beta|1\rangle) = |\phi\rangle\,.$$

(6 marks)

(Total = 20 marks)

QUESTION 3

Consider a one-dimensional harmonic oscillator with mass m and frequency ω, i.e., a particle with Hamiltonian

$$H_0 = \frac{P^2}{2m} + \frac{1}{2}m\omega^2 X^2 = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right)$$

where the annihilation operator is given by

$$a = \sqrt{\frac{m\omega}{2\hbar}}\left(X + i\frac{P}{m\omega}\right).$$

The oscillator is initially, at time $t = 0$, in the ground state $|0\rangle$. At time $t = 0$, a parametric interaction is turned on, which changes the Hamiltonian to

$$H = \frac{P^2}{2m} + \frac{1}{2}m\omega^2\left(1 + g\cos(2\omega t)\right)X^2$$

The parametric interaction modulates the resonant frequency at twice the resonant fre-quency. The strength of the modulation is characterized by the dimensionless constantg.The parametric interaction can be treated as a perturbation,

$$W(t) = \frac{1}{2}gm\omega^2 X^2\cos(2\omega t)$$

on top of the unperturbed Hamiltonian H_0. The state of the system at time t can be written in the energy-eigenstate (number) basis as

$$|\Psi(t)\rangle = \sum_{n=0}^{\infty} c_n(t)\,|n\rangle = e^{-i\omega t/2}\sum_{n=0}^{\infty} b_n(t)e^{-in\omega t}\,|n\rangle$$

We are interested in the state of the oscillator after the parametric interaction has been on for a time τ. Throughout the problem we are only interested in the secular (resonant) terms. These secular terms do not oscillate at frequency ω, and they dominate when $\omega\tau \gg 1$.

(a) Find the first-order secular contribution to the number-state amplitudes, i.e $b_n^{(1)}(\tau)$.

(5 marks)

(b) Find the second-order secular contribution to the number-state amplitudes, i.e $b_n^{(2)}(\tau)$.

(5 marks)

(c) Go to an interaction picture relative to H_0. Find the interaction-picture Hamiltonian, make the rotating-wave approximation (i.e., keep only resonant terms), and integrate the result to find the interaction-picture evolution operator $U_I(\tau, 0)$. Use this result to find the secular contributions to the amplitudes $b_n^{(1)}(t)$ and $b_n^{(2)}(t)$ to the same order as in parts (a) and (b). Your results should agree with those of parts (a) and (b).

(10 marks)

(Total = 20 marks)

QUESTION 4

Consider the scattering of a particle by an impenetrable sphere

$$V(r) = \begin{cases} \infty & \text{for } r < a \\ -V_0 & \text{for } a < r < b \\ 0 & \text{for } r > b \end{cases}$$

where $V_0 > 0$.

(a) Derive an expression for the s-wave phase shift (δ_0).

(8 marks)

(b) Estimate the total cross section in the low-energy limit if you are not near a resonance. You may assume without proof:

$$\frac{d\sigma}{d\Omega} = |f(\theta)|^2$$

$$f(\theta) = \left(\frac{1}{k}\right) \sum_{\ell=0}^{\infty} (2\ell + 1) e^{i\delta_\ell} \sin \delta_\ell P_\ell(\cos \theta)$$

(8 marks)

(c) For what value of V_0 do resonances occur?

(4 marks)

(Total = 20 marks)

QUESTION 5

Consider the field theory of a complex-valued scalar field obeying the Klein-Gordon equation. The action of this theory is

$$S = \int d^4x \left(\partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi\right)$$

(a) Find the conjugate momenta to $\phi(x)$ and $\phi^*(x)$ and the canonical commutation relations. Show that the Hamiltonian is

$$H = \int d^3x(\pi^* \pi + \Delta\phi^* \cdot \Delta\phi + m^2 \phi^* \phi)$$

(10 marks)

(b) Diagonalize H by introducing creation and annihilation operators. Show that the theory contains two sets of particles of mass m.

(10 marks)

(Total = 20 marks)

General Relations

- $e^{\hat{A}}e^{\hat{B}} = e^{\hat{A}+\hat{B}}e^{[\hat{A},\hat{B}]/2}$
- $e^{\hat{A}}\hat{B}e^{-\hat{A}} = \hat{B} + [\hat{A},\hat{B}] + \frac{1}{2!}[\hat{A},[\hat{A},\hat{B}]] + \frac{1}{3!}[\hat{A},[\hat{A},[\hat{A},\hat{B}]]] + \dots$
- $e^{i\theta a^\dagger a} a e^{-i\theta a^\dagger a} = a e^{-i\theta}$
- $[\hat{x},\hat{p}] = i\hbar$
- $[a, a^\dagger] = 1$
- $a\,|n\rangle = \sqrt{n}\,|n-1\rangle$
- $a^\dagger\,|n\rangle = \sqrt{n+1}\,|n+1\rangle$
- $a^\dagger a\,|n\rangle = n\,|n\rangle$
- $\Delta x \Delta p \leq \frac{\hbar}{2}$, $\Delta E \Delta t \leq \frac{\hbar}{2}$
- $\hat{A}\,|\psi_n\rangle = a_n\,|\psi_n\rangle$, $P_n(a_n) = \frac{|\langle\psi_n|\psi\rangle|^2}{\langle\psi|\psi\rangle}$
- $\hat{p} = i\sqrt{\frac{m\hbar\omega}{2}}(a^\dagger - a)$
- $\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(a + a^\dagger)$
- $\langle x|\psi\rangle = \frac{1}{(2\pi)^{3/2}}\sum i^\ell(2\ell+1)A_\ell(r)P_\ell(\cos\theta)$, $A_\ell = c_\ell^{(1)}h_\ell^{(1)}(kr) + c_\ell^{(2)}h_\ell^{(2)}(kr)$
- $h_\ell^{(1)} = j_\ell + in_\ell$, $h_\ell^{(2)} = j_\ell - in_\ell$
- $j_\ell(x) = (-x)^\ell\left(\frac{1}{x}\frac{d}{dx}\right)^\ell\frac{\sin x}{x}$, $n_\ell(x) = -(-x)^\ell\left(\frac{1}{x}\frac{d}{dx}\right)^\ell\frac{\cos x}{x}$