UNIVERSITY OF SWAZILAND

FACULTY OF SCIENCE

DEPARTMENT OF PHYSICS

SUPPLEMENTARY EXAMINATION 2018/2019

TITLE OF PAPER	:	MATHEMATICAL METHODS FOR PHYSICISTS
COURSE NUMBER	:	P272/PHY271
TIME ALLOWED	:	THREE HOURS
INSTRUCTIONS	:	ANSWER ANY FOUR OUT OF FIVE QUESTIONS. EACH QUESTION CARRIES 25 MARKS. MARKS FOR DIFFERENT SECTIONS ARE SHOWN IN THE RIGHT-HAND MARGIN.

THIS PAPER HAS EIGHT PAGES, INCLUDING THIS PAGE.

DO NOT OPEN THE PAPER UNTIL PERMISSION HAS BEEN GIVEN BY THE INVIGILATOR.

P272/PHY271 MATHEMATICAL METHODS FOR PHYSICIST

Question one

A vector field $\vec{F}$ expressed in cylindrical coordinates is given as
$\vec{F} = \vec{e}_\rho \left(7\,\rho^2 \cos(\phi)\right) + \vec{e}_\phi \left(9\,\rho^2\right) + \vec{e}_z \left(5\,\rho\, z\right)$.

(a) Evaluate the value of $\oint_L \vec{F} \bullet d\vec{l}$ if L is the circular closed loop of radius *3* on $z = 2$ plane in counter clockwise sense and centered at $\rho = 0$ & $z = 2$, i.e.,
$L : \left(\rho = 3 \;,\; 0 \le \phi \le 2\pi \;,\; z = 2 \quad \& \quad d\vec{l} = +\vec{e}_\phi\, \rho\, d\phi \xrightarrow{\rho = 3} \vec{e}_\phi\, 3\, d\phi \right)$

(7 marks)

(b) (i) Evaluate $\vec{\nabla} \times \vec{F}$ and show that
$\vec{\nabla} \times \vec{F} = \vec{e}_\phi \left(-5\,z\right) + \vec{e}_z \left(27\,\rho + 7\,\rho \sin(\phi)\right)$ **(6 marks)**

(ii) Evaluate the value of $\iint_S \left(\vec{\nabla} \times \vec{F}\right) \bullet d\vec{s}$ where S is bounded by L given in *(a)* , i.e.,
$S : \left(0 \le \rho \le 3 \;,\; 0 \le \phi \le 2\pi \;,\; z = 2 \quad \& \quad d\vec{s} = \vec{e}_z\, \rho\, d\rho\, d\phi\right)$
Compare this value with that obtained in *(a)* and make a brief comment.

(6+1 marks)

(c) Show that the given vector field satisfies the following vector identity that
$\vec{\nabla} \bullet \left(\vec{\nabla} \times \vec{F}\right) \equiv 0$ **(5 marks)**

Question two

(a) Given a scalar function $f = 5\, r^4 \sin(\theta)\sin(\phi)$ in spherical coordinates, by direct substitution show that it satisfies the following vector identity that $\vec{\nabla} \times \left(\vec{\nabla} f\right) \equiv 0$. **(8 marks)**

(b) Given the following differential equation as :

$$\frac{d^2\, y(x)}{d\, x^2} - \frac{d\, y(x)}{d\, x} - 2\, y(x) = 0 \quad .$$

Solve by using the power series method , i.e., setting

$$y(x) = \sum_{n=0}^{\infty} a_n\, x^{n+s} \quad and \quad a_0 \xrightarrow{\text{set as}} 1$$

(i) Write down the indicial equations and recurrence relations. Deduce that

$$s = 0 \;\; or \;\; 1 \;\; \text{and} \quad a_1 = \begin{cases} 1 & for \quad s = 0 \\ \dfrac{1}{2} & for \quad s = 1 \end{cases} .$$

(10 marks)

(ii) For $s = 0$ case, set $a_0 = 1$ and use the recurrence relation to calculate the values of a_n up to the value of a_5 . Then write down this independent solution in its power series form truncated up to a_5 term. **(7 marks)**

Question three

Given the following non-homogeneous differential equation as

$$\frac{d\,x(t)}{d\,t} - 4\,x(t) = 10\,\sin(2t) \quad \cdots\cdots \quad (1)$$

and also given the following initial condition that $x(0) = 6$,

(a) (i) find its particular solution $x_p(t)$ and show that

$x_p(t) = -2\,\sin(2t) - \cos(2t)$ **(5 marks)**

(ii) For the homogeneous part of the given non-homogeneous differential equation , i.e., $\frac{d\,x(t)}{d\,t} - 4\,x(t) = 0$, by direct substitution show that its general solution is $x_h(t) = k\,e^{4t}$ where k is an arbitrary constant.

(2 marks)

(iii) Write down the general solution of the given non-homogeneous differential equation in terms of the answers obtained in *(a)(i) & (a)(ii)* . From the given initial condition, i.e., $x(0) = 6$, find its specific solution $x_s(t)$ and show that

$x_s(t) = 7\,e^{4t} - 2\,\sin(2t) - \cos(2t)$. **(1+2 marks)**

(b) (i) Find the Laplace transform of $x(t)$, i.e., $L\{x(t)\} \xrightarrow{set\ as} X(s)$, from the above given non-homogeneous differential equation and initial condition, deduce that

$X(s) = \frac{6}{s-4} + \frac{20}{(s-4)(s^2+4)}$ **(6 marks)**

(ii) Convert $X(s)$ in *(b)(i)* into its partial fraction and show that

$X(s) = \frac{7}{s-4} - \frac{s}{s^2+4} - \frac{4}{s^2+4}$ **(6 marks)**

(Hint : $\frac{20}{(s-4)(s^2+4)} \xrightarrow{set\ as} \frac{k_1}{s-4} + \frac{k_2\,s}{s^2+4} + \frac{k_3}{s^2+4}$)

(iii) Take the inverse Laplace transform of $X(s)$ in *(b)(ii)* to find the specific solution of $x(t)$. **(3 marks)**

Question four

The longitudinal vibration amplitude $u(x,t)$ of a given vibrating string of length 10 meters , fixed at its two ends ,i.e., $u(0,t)=0$ & $u(10,t)=0$, satisfies the following 1-D wave equation

$$\frac{\partial^2 u(x,t)}{\partial t^2} = 25\,\frac{\partial^2 u(x,t)}{\partial x^2} \quad \cdots\cdots \quad (1)$$

(a) set $u(x,t) = F(x)\,G(t)$ and apply the techniques of separation of variables to deduce the following two ordinary differential equations that

$$\begin{cases} \dfrac{d^2 F(x)}{d x^2} = \dfrac{k}{25}\,F(x) & \cdots\cdots \quad (2) \\ \dfrac{d^2 G(t)}{d t^2} = k\,G(t) & \cdots\cdots \quad (3) \end{cases}$$

where k is a separation constant.
For our given problem, k needs to be any negative constant, explain briefly why ?

(4 + 2 marks)

(b) Consider the following $u(x,t)$ that

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) \qquad where \quad u_n(x,t) = E_n \sin\left(\frac{n\pi x}{10}\right)\cos\left(\frac{n\pi t}{2}\right) ,$$

(i) by direct substitution, show that $u_n(x,t)$ satisfies the given 1-D wave equation eq.(1) , **(4 marks)**

(ii) show that $u_n(x,t)$ satisfies the two fixed conditions , i.e., $u_n(0,t)=0$ & $u_n(10,t)=0$, **(2 marks)**

(iii) show that $u_n(x,t)$ satisfies the zero initial speed condition , i.e.,

$$\left.\frac{\partial u_n(x,t)}{\partial t}\right|_{t=0} = 0 ,$$ **(3 marks)**

(iv) if the initial position of the vibrating string, i.e., $u(x,0)$, is given as

$$u(x,0) = \begin{cases} 2x & for \quad 0 \le x \le 6 \\ -3x+30 & for \quad 6 \le x \le 10 \end{cases} ,$$

find the values of E_n and show that

$$E_n = \frac{100}{n^2\pi^2}\sin\left(\frac{3n\pi}{5}\right) \quad where \quad n = 1,2,3,\cdots$$

Also calculate the value of E_1 . **(9 + 1 marks)**

Question five

Given the following equations for coupled oscillator system as :

$$\begin{cases} \dfrac{d^2 x_1(t)}{dt^2} = -14\, x_1(t) + 4\, x_2(t) \\ \dfrac{d^2 x_2(t)}{dt^2} = 5\, x_1(t) - 6\, x_2(t) \end{cases}$$

(a) set $x_1(t) = X_1\, e^{i\omega t}$ & $x_2(t) = X_2\, e^{i\omega t}$, deduce the following matrix equation

$A\,X = -\,\omega^2\,X$ where $A = \begin{pmatrix} -14 & 4 \\ 5 & -6 \end{pmatrix}$ & $X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$ **(4 marks)**

(b) find the eigenfrequencies ω of the given coupled system , **(5 marks)**

(c) find the eigenvectors X of the given coupled system corresponding to each eigenfrequencies found in (b). **(6 marks)**

(d) find the normal coordinates for the given coupled system, **(7 marks)**

(e) write down the general solution of the given system. **(3 marks)**

Useful informations

The transformations between rectangular and spherical coordinate systems are :

$$\begin{cases} x = r\sin(\theta)\cos(\phi) \\ y = r\sin(\theta)\sin(\phi) \\ z = r\cos(\theta) \end{cases} \quad \& \quad \begin{cases} r = \sqrt{x^2+y^2+z^2} \\ \theta = \tan^{-1}\left(\dfrac{\sqrt{x^2+y^2}}{z}\right) \\ \phi = \tan^{-1}\left(\dfrac{y}{x}\right) \end{cases}$$

The transformations between rectangular and cylindrical coordinate systems are :

$$\begin{cases} x = \rho\cos(\phi) \\ y = \rho\sin(\phi) \\ z = z \end{cases} \quad \& \quad \begin{cases} \rho = \sqrt{x^2+y^2} \\ \phi = \tan^{-1}\left(\dfrac{y}{x}\right) \\ z = z \end{cases}$$

$$\vec{\nabla} f = \vec{e}_1 \frac{1}{h_1}\frac{\partial f}{\partial u_1} + \vec{e}_2 \frac{1}{h_2}\frac{\partial f}{\partial u_2} + \vec{e}_3 \frac{1}{h_3}\frac{\partial f}{\partial u_3}$$

$$\vec{\nabla} \bullet \vec{F} = \frac{1}{h_1 h_2 h_3}\left(\frac{\partial(F_1 h_2 h_3)}{\partial u_1} + \frac{\partial(F_2 h_1 h_3)}{\partial u_2} + \frac{\partial(F_3 h_1 h_2)}{\partial u_3}\right)$$

$$\vec{\nabla} \times \vec{F} = \frac{\vec{e}_1}{h_2 h_3}\left(\frac{\partial(F_3 h_3)}{\partial u_2} - \frac{\partial(F_2 h_2)}{\partial u_3}\right) + \frac{\vec{e}_2}{h_1 h_3}\left(\frac{\partial(F_1 h_1)}{\partial u_3} - \frac{\partial(F_3 h_3)}{\partial u_1}\right) + \frac{\vec{e}_3}{h_1 h_2}\left(\frac{\partial(F_2 h_2)}{\partial u_1} - \frac{\partial(F_1 h_1)}{\partial u_2}\right)$$

where $\vec{F} = \vec{e}_1 F_1 + \vec{e}_2 F_2 + \vec{e}_3 F_3$ and

(u_1, u_2, u_3)	represents	(x, y, z)	for rectangular coordinate system
	represents	(ρ, ϕ, z)	for cylindrical coordinate system
	represents	(r, θ, ϕ)	for spherical coordinate system
$(\vec{e}_1, \vec{e}_2, \vec{e}_3)$	represents	$(\vec{e}_x, \vec{e}_y, \vec{e}_z)$	for rectangular coordinate system
	represents	$(\vec{e}_\rho, \vec{e}_\phi, \vec{e}_z)$	for cylindrical coordinate system
	represents	$(\vec{e}_r, \vec{e}_\theta, \vec{e}_\phi)$	for spherical coordinate system
(h_1, h_2, h_3)	represents	$(1, 1, 1)$	for rectangular coordinate system
	represents	$(1, \rho, 1)$	for cylindrical coordinate system
	represents	$(1, r, r\sin(\theta))$	for spherical coordinate system

$$f(t) = f(t+2L) = f(t+4L) = \cdots = \sum_{n=0}^{\infty} a_n \cos\left(\frac{n\pi t}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi t}{L}\right) \quad \textit{where}$$

$$a_0 = \frac{1}{2L}\int_0^{2L} f(t)\,dt \;,\; a_n = \frac{1}{L}\int_0^{2L} f(t)\cos\left(\frac{n\pi t}{L}\right)dt \;\&\; b_n = \frac{1}{L}\int_0^{2L} f(t)\sin\left(\frac{n\pi t}{L}\right)dt \quad \textit{for } n = 1,2,3,\cdots$$

$$\int (t\sin(kt))\,dt = -\frac{t\cos(kt)}{k} + \frac{\sin(kt)}{k^2}$$

$$\int (t\cos(kt))\,dt = \frac{t\sin(kt)}{k} + \frac{\cos(kt)}{k^2}$$

$F(s) \equiv L\{f(t)\}$	$f(t)$
$1/s$ $1/s^2$ $1/s^n \quad n=1,2,3,\cdots$	1 t $t^{n-1}/(n-1)! \quad n=1,2,3,\cdots$
$1/(s-a)$ $1/(s-a)^2$ $1/(s-a)^n \quad n=1,2,3,\cdots$ $F(s-a)$	e^{at} $e^{at}\, t$ $e^{at}\, t^{n-1}/(n-1)! \quad n=1,2,3,\cdots$ $e^{at}\, f(t) \qquad s-shift\ theory$
$1/(s^2+\omega^2)$ $s/(s^2+\omega^2)$ $1/(s^2-a^2)$ $s/(s^2-a^2)$	$\sin(\omega t)/\omega$ $\cos(\omega t)$ $\sinh(at)/a$ $\cosh(at)$
$e^{-as}/s \quad a>0$ $e^{-as}\, F(s) \quad a>0$	$u(t-a) \quad a>0$ $f(t-a)\, u(t-a) \qquad t-shift\ theory$
$F(s)\, G(s)$	$h(t) = (f * g)(t) = \int_{\tau=0}^{t} f(\tau)\, g(t-\tau)\, d\tau$ $= \int_{\tau=0}^{t} f(t-\tau)\, g(\tau)\, d\tau \quad convolution\ theory$

where $u(t-a) \equiv \begin{cases} 0 & if \quad t<a \\ 1 & if \quad t>a \end{cases}$ *is an unitary step function & its Laplace transform is*

$$L\{u(t-a)\} = \int_{t=0}^{\infty} u(t-a)\, e^{-st}\, dt \quad where \quad s>0$$

$$= \int_{t=0}^{a} (0)\, e^{-st}\, dt + \int_{t=a}^{\infty} (1)\, e^{-st}\, dt$$

$$= 0 + \left(-\frac{1}{s} e^{-st}\right)\Bigg|_{t=a}^{\infty} = \left(-\frac{1}{s} e^{-st}\right)\Bigg|_{t=0}^{\infty} = (0) - \left(-\frac{1}{s} e^{-sa}\right) = \frac{1}{s} e^{-sa}$$

$$L\{y'(t)\} = -\, y(0) + s\, L\{y(t)\}$$

$$L\{y''(t)\} = -\, y'(0) - s\, y(0) + s^2\, L\{y(t)\}$$