UNIVERSITY OF SWAZILAND

FACULTY OF SCIENCE AND ENGINEERING

DEPARTMENT OF PHYSICS

MAIN EXAMINATION 2017/2018

TITLE OF PAPER	:	MATHEMATICAL METHODS FOR PHYSICISTS
COURSE NUMBER	:	P272/PHY271
TIME ALLOWED	:	THREE HOURS
INSTRUCTIONS	:	ANSWER ANY FOUR OUT OF FIVE QUESTIONS. EACH QUESTION CARRIES 25 MARKS. MARKS FOR DIFFERENT SECTIONS ARE SHOWN IN THE RIGHT-HAND MARGIN.

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P272/PHY271 MATHEMATICAL METHODS FOR PHYSICIST

Question one

(a) Given a scalar function in spherical coordinates as $f = r^2 \sin(\theta)\cos(\phi)$,

(i) find the value of $\bar{\nabla} f$ at a point $P(3, 90^0, 45^0)$, **(4 marks)**

(ii) find the value of the directional derivative of f at a point $P(3, 90^0, 45^0)$ along the direction of $\vec{A}\ (= \vec{e}_r\, 2 + \vec{e}_\theta + \vec{e}_\phi\, 2)$.

(3 marks)

(b) Given a vector field in Cartesian coordinates as $\bar{F} = \bar{e}_x\,(y^2) + \bar{e}_y\,(2xy) + \bar{e}_z\,(3z^2)$,

find the value of the line integral $\int_{P_1, L}^{P_2} \bar{F} \bullet d\bar{l}$ if $P_1 : (1, 3, 0)$, $P_2 : (3, 11, 0)$

and

(i) L : a straight line from P_1 to P_2 on $x-y$ plane , i.e., $z = 0$ plane .

(6 marks)

(ii) L : a parabolic path described by $y = x^2 + 2$ from P_1 to P_2 on $x-y$ plane . Compare this answer with that obtained in *(b)(i)* and comment on whether the given $\bar{F}$ is a conservative vector field or not. **(7+1 marks)**

(iii) Find $\bar{\nabla} \times \bar{F}$. Does this result agree with the comment you made in *(b)(ii)*?

(3+1 marks)

Question two

A vector field $\bar{F}$ expressed in cylindrical coordinates is given as
$\bar{F} = \bar{e}_\rho \left(3\,\rho^2\right) + \bar{e}_\phi \left(2\,\rho\, z\right) + \bar{e}_z \left(z^2 \cos(\phi)\right)$.

(a) (i) Evaluate the value of $\oint_L \bar{F} \bullet d\bar{l}$ if L is the circular closed loop of radius 5 on $z = 2$ plane in counter clockwise sense and centered at $\rho = 0$ & $z = 2$, i.e.,
$L : \left(\rho = 5\ ,\ 0 \le \phi \le 2\pi\ ,\ z = 2\ \ \&\ \ d\bar{l} = +\bar{e}_\phi\, \rho\, d\phi \xrightarrow{\rho=5} \bar{e}_\phi\, 5\, d\phi\right)$

(7 marks)

(ii) Evaluate the value of $\iint_S \left(\bar{\nabla} \times \bar{F}\right) \bullet d\bar{s}$ where S is bounded by L given in *(a)(i)* , i.e.,
$S : \left(0 \le \rho \le 5\ ,\ 0 \le \phi \le 2\pi,\ z = 2\ \ \&\ \ d\bar{s} = \bar{e}_z\, \rho\, d\rho\, d\phi\right)$
Compare this value with that obtained in (a)(i) and make a brief comment.

(12+1 marks)

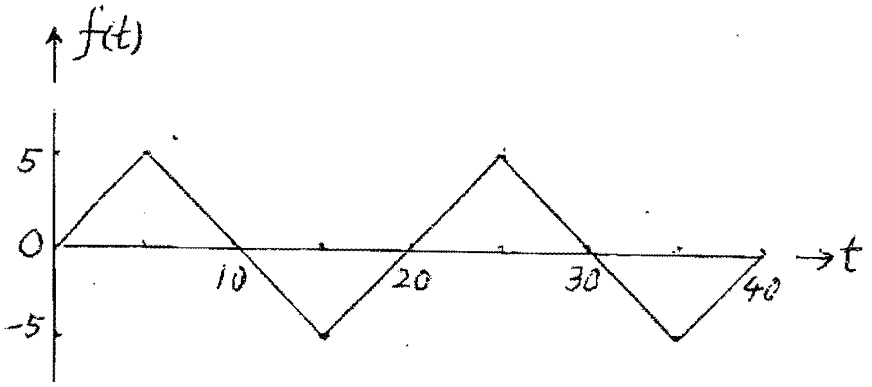
(b) Show that the given vector field satisfies the following vector identity that
$\bar{\nabla} \bullet \left(\bar{\nabla} \times \bar{F}\right) \equiv 0$ **(5 marks)**

Question three

Given the following non-homogeneous differential equation as $\frac{d^2 x(t)}{dt^2} + 5\,x(t) = f(t)$, where $f(t)$ is a periodic driving force of period 20 , i.e., $f(t) = f(t+20) = f(t+40) = \cdots$., and its first period description is

$$f(t) = \begin{cases} t & for \quad 0 \le t \le 5 \\ -t+10 & for \quad 5 \le t \le 15 \\ t-20 & for \quad 15 \le t \le 20 \\ 0 & for \quad t \ge 20 \end{cases}$$

and is plotted for the first two period, i.e., for $0 \le t \le 40$, as the diagram below



(a) Set $f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi t}{10}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi t}{10}\right)$.

(i) One can conclude without calculation that $a_n = 0 \quad \forall \quad n = 0,1,2,\cdots$ based on a special character of our given $f(t)$. What is that special character? **(1 marks)**

(ii) Find the Fourier sine series coefficients b_n and show that

$$b_n = \frac{20\left(\sin\left(\frac{n\pi}{2}\right) - \sin\left(\frac{3n\pi}{2}\right)\right)}{n^2 \pi^2} \qquad \forall \quad n = 1,2,3,\cdots$$ **(12 marks)**

Thus the Fourier series representation of the given periodic function is

$$f(t) = \sum_{n=1}^{\infty} \frac{20\left(\sin\left(\frac{n\pi}{2}\right) - \sin\left(\frac{3n\pi}{2}\right)\right)}{n^2 \pi^2} \sin\left(\frac{n\pi t}{10}\right) \quad \cdots\cdots \quad (1)$$

(b) Find the particular solution of the given non-homogeneous differential equation $x_p(t)$ and show that

$$x_p(t) = \sum_{n=1}^{\infty} \left\{ -\frac{2000\left(\sin\left(\frac{n\pi}{2}\right) - \sin\left(\frac{3n\pi}{2}\right)\right)}{n^2 \pi^2 \left(n^2 \pi^2 - 500\right)} \sin\left(\frac{n\pi t}{10}\right) \right\}$$ **(12 marks)**

Question four

(a) Given the following 2-D Laplace equation in spherical coordinates as

$$\nabla^2 f(r,\theta) = 0 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial f(r,\theta)}{\partial r}\right) + \frac{1}{r^2 \sin(\theta)}\frac{\partial}{\partial \theta}\left(\sin(\theta)\frac{\partial f(r,\theta)}{\partial \theta}\right) .$$

(i) Set $f(r,\theta) = F(r)\,G(\theta)$ and use separation variable scheme to separate the above partial differential equation into the following two ordinary differential equations.

$$\begin{cases} \dfrac{d}{dr}\left(r^2 \dfrac{d(F(r))}{dr}\right) = k\,F(r) & \cdots\cdots \quad (1) \\ \dfrac{1}{\sin(\theta)}\dfrac{d}{d\theta}\left(\sin(\theta)\dfrac{d(G(\theta))}{d\theta}\right) + k\,G(\theta) = 0 & \cdots\cdots \quad (2) \end{cases}$$

where k is a separation constant. **(5 marks)**

(ii) Set $x \equiv \cos(\theta)$ & $G(\theta) \equiv y(x)$, show that *eq.(2)* can be transformed to the following differential equation :

$$\left(1 - x^2\right)\frac{d^2 y(x)}{dx^2} - 2x\frac{d\,y(x)}{dx} + k\,y(x) = 0 \quad \cdots\cdots \quad (3)$$ **(3 marks)**

(b) If $k = 12$, *eq.(3)* in *(a)(ii)* becomes $\left(1 - x^2\right)\dfrac{d^2 y(x)}{dx^2} - 2x\dfrac{d\,y(x)}{dx} + 12\,y(x) = 0$.

Set $y(x) = \sum_{n=0}^{\infty} a_n\, x^{n+s}$ & $a_0 \neq 0$ and utilize the power series method,

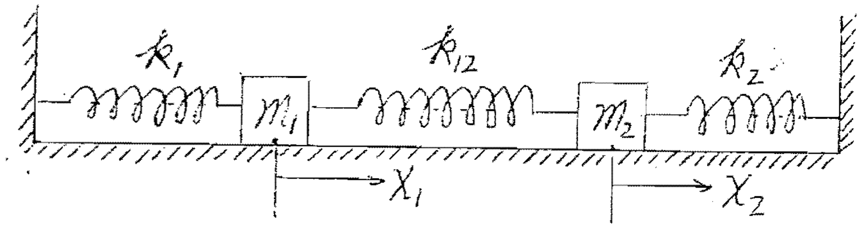
(i) write down its indicial equations and show that $s = 0$ *or* 1 and $a_1 = 0$. **(7 marks)**

(ii) For $s = 1$ independent solution, named as $y_2(x)$, write down its recurrence relation. Set $a_0 = 1$ and use the recurrence relation to generate $y_2(x)$ in power series form truncated up to a_6 term. **(7 marks)**

(iii) Show that $y_2(x)$ is linearly dependent to one of the well-known Legendre polynomial $P_3(x)\left(\equiv \dfrac{5}{2}x^3 - \dfrac{3}{2}x\right)$. **(3 marks)**

Question five

Two simple harmonic oscillators are joined by a spring with a spring constant k_{12} as shown in the diagram below :



The equations of motion for this coupled oscillator system ignoring friction are given as

$$\begin{cases} m_1 \dfrac{d^2 x_1(t)}{dt^2} = -\left(k_1 + k_{12}\right) x_1(t) + k_{12}\, x_2(t) \\ m_2 \dfrac{d^2 x_2(t)}{dt^2} = k_{12}\, x_1(t) - \left(k_2 + k_{12}\right) x_2(t) \end{cases}$$

where x_1 & x_2 are horizontal displacements of m_1 & m_2 measured from their respective resting positions.

If given $m_1 = 3\ kg$, $m_2 = 6\ kg$, $k_1 = 6\ \frac{N}{m}$, $k_2 = 12\ \frac{N}{m}$ & $k_{12} = 6\ \frac{N}{m}$.

(a) Set $x_1(t) = X_1\, e^{i\omega t}$ & $x_2(t) = X_2\, e^{i\omega t}$. Then the above given equations can be deduced to the following matrix equation $A\,X = -\,\omega^2\, X$ where

$$A = \begin{pmatrix} -4 & 2 \\ 1 & -3 \end{pmatrix} \quad \& \quad X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} .$$ **(5 marks)**

(b) Find the eigenfrequencies ω of the given coupled system . **(6 marks)**

(c) Find the eigenvectors X of the given coupled system corresponding to each eigenfrequencies found in *(b)* . **(6 marks)**

(d) Write down the general solutions for $x_1(t)$ & $x_2(t)$. **(2 marks)**

(e) Find the specific solutions for $x_1(t)$ & $x_2(t)$ if the initial conditions are given as

$$x_1(0) = 1\ ,\ x_2(0) = -2\ ,\ \left.\frac{d x_1(t)}{dt}\right|_{t=0} = 0 \quad \& \quad \left.\frac{d x_2(t)}{dt}\right|_{t=0} = 0\ .$$ **(6 marks)**

Useful informations

The transformations between rectangular and spherical coordinate systems are :

$$\begin{cases} x = r\sin(\theta)\cos(\phi) \\ y = r\sin(\theta)\sin(\phi) \\ z = r\cos(\theta) \end{cases} \quad \& \quad \begin{cases} r = \sqrt{x^2 + y^2 + z^2} \\ \theta = \tan^{-1}\left(\dfrac{\sqrt{x^2 + y^2}}{z}\right) \\ \phi = \tan^{-1}\left(\dfrac{y}{x}\right) \end{cases}$$

The transformations between rectangular and cylindrical coordinate systems are :

$$\begin{cases} x = \rho\cos(\phi) \\ y = \rho\sin(\phi) \\ z = z \end{cases} \quad \& \quad \begin{cases} \rho = \sqrt{x^2 + y^2} \\ \phi = \tan^{-1}\left(\dfrac{y}{x}\right) \\ z = z \end{cases}$$

$$\bar{\nabla} f = \bar{e}_1 \frac{1}{h_1}\frac{\partial f}{\partial u_1} + \bar{e}_2 \frac{1}{h_2}\frac{\partial f}{\partial u_2} + \bar{e}_3 \frac{1}{h_3}\frac{\partial f}{\partial u_3}$$

$$\bar{\nabla} \bullet \bar{F} = \frac{1}{h_1 h_2 h_3}\left(\frac{\partial(F_1 h_2 h_3)}{\partial u_1} + \frac{\partial(F_2 h_1 h_3)}{\partial u_2} + \frac{\partial(F_3 h_1 h_2)}{\partial u_3}\right)$$

$$\bar{\nabla} \times \bar{F} = \frac{\bar{e}_1}{h_2 h_3}\left(\frac{\partial(F_3 h_3)}{\partial u_2} - \frac{\partial(F_2 h_2)}{\partial u_3}\right) + \frac{\bar{e}_2}{h_1 h_3}\left(\frac{\partial(F_1 h_1)}{\partial u_3} - \frac{\partial(F_3 h_3)}{\partial u_1}\right) + \frac{\bar{e}_3}{h_1 h_2}\left(\frac{\partial(F_2 h_2)}{\partial u_1} - \frac{\partial(F_1 h_1)}{\partial u_2}\right)$$

where $\bar{F} = \bar{e}_1 F_1 + \bar{e}_2 F_2 + \bar{e}_3 F_3$ and

(u_1, u_2, u_3)	represents	(x, y, z)	for rectangular coordinate system
	represents	(ρ, ϕ, z)	for cylindrical coordinate system
	represents	(r, θ, ϕ)	for spherical coordinate system
$(\bar{e}_1, \bar{e}_2, \bar{e}_3)$	represents	$(\bar{e}_x, \bar{e}_y, \bar{e}_z)$	for rectangular coordinate system
	represents	$(\bar{e}_\rho, \bar{e}_\phi, \bar{e}_z)$	for cylindrical coordinate system
	represents	$(\bar{e}_r, \bar{e}_\theta, \bar{e}_\phi)$	for spherical coordinate system
(h_1, h_2, h_3)	represents	$(1, 1, 1)$	for rectangular coordinate system
	represents	$(1, \rho, 1)$	for cylindrical coordinate system
	represents	$(1, r, r\sin(\theta))$	for spherical coordinate system

$$f(t) = f(t + 2L) = f(t + 4L) = \cdots = \sum_{n=0}^{\infty} a_n \cos\left(\frac{n\pi t}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi t}{L}\right) \quad where$$

$$a_0 = \frac{1}{2L}\int_0^{2L} f(t)\,dt \;,\; a_n = \frac{1}{L}\int_0^{2L} f(t)\cos\left(\frac{n\pi t}{L}\right)dt \;\&\; b_n = \frac{1}{L}\int_0^{2L} f(t)\sin\left(\frac{n\pi t}{L}\right)dt \;\; for \;\; n = 1,2,3,\cdots$$

$$\int (t\sin(kt))\,dt = -\frac{t\cos(kt)}{k} + \frac{\sin(kt)}{k^2}$$

$$\int (t\cos(kt))\,dt = \frac{t\sin(kt)}{k} + \frac{\cos(kt)}{k^2}$$