UNIVERSITY OF SWAZILAND

FACULTY OF SCIENCE AND ENGINEERING

DEPARTMENT OF PHYSICS

MAIN EXAMINATION 2015/2016

TITLE OF PAPER	:	ELECTROMAGNETIC THEORY I
COURSE NUMBER	:	P331
TIME ALLOWED	:	THREE HOURS
INSTRUCTIONS	:	ANSWER ANY FOUR OUT OF FIVE QUESTIONS. EACH QUESTION CARRIES 25 MARKS. MARKS FOR DIFFERENT SECTIONS ARE SHOWN IN THE RIGHT-HAND MARGIN.

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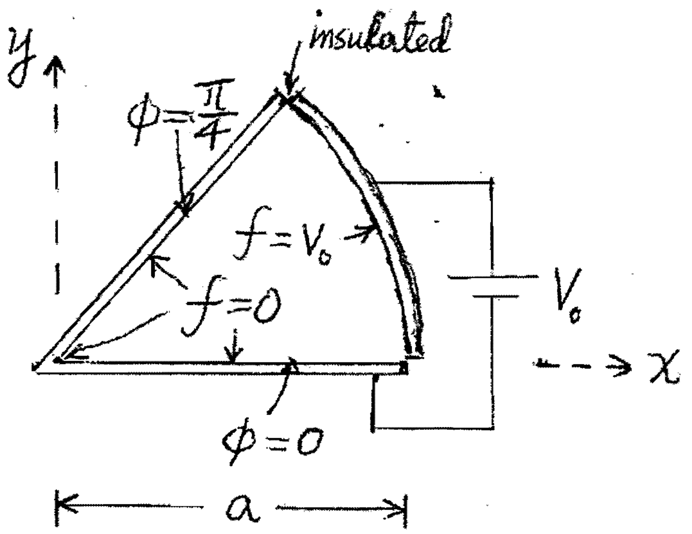
P331 Electromagnetic Theory I

Question one

A V – tube capacitor is extended very long into z direction with its cross section as shown below:



The electric potential $f(\rho,\phi)$ in cylindrical coordinates for the region between two conductors, i.e., $0 \le \rho \le a$ & $0 \le \phi \le \frac{\pi}{4}$, satisfies the following two dimensional Laplace equation :

$$\rho \frac{\partial\left(\rho \frac{\partial f(\rho,\phi)}{\partial \rho}\right)}{\partial \rho} + \frac{\partial^2 f(\rho,\phi)}{\partial \phi^2} = 0$$

(a) (i) Set $f(\rho,\phi) = F(\rho)\, G(\phi)$ and use separation of variables to deduce the following two ordinary differential equations :

$$\begin{cases} \rho \dfrac{d\left(\rho \dfrac{dF(\rho)}{d\rho}\right)}{d\rho} = -k\, F(\rho) & \cdots\cdots \quad (1) \\ \dfrac{d^2 G(\phi)}{d\phi^2} = k\, G(\phi) & \cdots\cdots \quad (2) \end{cases}$$

where k is a separation constant. **(4 marks)**

(ii) Based on eq.(2), i.e., differential equation for ϕ, explain why the eigenvalues for k are $k = -m^2$ *where* $m = 1,2,3,\cdots\cdots$ **(3 marks)**

(iii) By direct substitution, show that ρ^m & ρ^{-m} are the two independent solution to eq.(1) with $k = -m^2$. **(3 marks)**

Question one (continued)

(b) The general solution for (a)(i) is

$$f(\rho,\phi)=\sum_{m=1}^{\infty} f_m(\rho,\phi)$$

$$=\sum_{m=1}^{\infty}\left(A_m\,\rho^{m}+B_m\,\rho^{-m}\right)\left(C_m\,\cos(m\phi)+D_m\,\sin(m\phi)\right)\quad\cdots\cdots\quad(3)$$

where A_m , B_m , C_m & D_m are arbitrary constants. This general solution is subjected to the following four boundary conditions :

$BC(1)$: $f_m(0,\phi)=0 \quad \forall \quad 0\le\phi\le\frac{\pi}{4}$

$BC(2)$: $f_m(\rho,0)=0 \quad \forall \quad 0\le\rho\le a$

$BC(3)$: $f_m(\rho,\frac{\pi}{4})=0 \quad \forall \quad 0\le\rho\le a$

$BC(4)$: $f(a,\phi)=V_0 \quad \forall \quad 0\le\phi\le\frac{\pi}{4}$

(i) Apply BC(1) and deduce from eq.(3) that

$$f(\rho,\phi)=\sum_{m=1}^{\infty}\left(A_m\,\rho^{m}\right)\left(C_m\,\cos(m\phi)+D_m\,\sin(m\phi)\right)\quad\cdots\cdots\quad(4)$$

(2 marks)

(ii) Apply BC(2) and deduce from eq.(4) that

$$f(\rho,\phi)=\sum_{m=1}^{\infty}\left(A_m\,\rho^{m}\right)\left(D_m\,\sin(m\phi)\right)\quad \textit{name}\quad \left(A_m\,D_m\right)\quad \textit{as}\quad E_m$$

$$=\sum_{m=1}^{\infty}\left(E_m\,\rho^{m}\,\sin(m\phi)\right)\quad\cdots\cdots\quad(5)$$

(2 marks)

(iii) Apply BC(3) and deduce from eq.(5) that

$$f(\rho,\phi)=\sum_{n=1}^{\infty}\left(F_n\,\rho^{4n}\,\sin(4n\phi)\right)\quad\cdots\cdots\quad(6)$$

where $F_n\equiv E_{4n}$ & $n=1,2,3,\cdots\cdots$

(3 marks)

(iv) Apply BC(4) and find the values of F_n in terms of V_0 , a & n and show that

$$F_n=\frac{2\,V_0\,\left(1-\cos(n\pi)\right)}{n\,\pi\,a^{4n}}\qquad n=1,2,3,\cdots\cdots$$

(8 marks)

(Hint : $\int_{\phi=0}^{\frac{\pi}{4}}\sin(4n\phi)\sin(4m\phi)\,d\phi=\begin{cases}0 & if\quad n\neq m\\ \frac{\pi}{8} & if\quad n=m\end{cases}$)

Question two

(a) A very long thin conducting wire situated along z-axis and carries a static current I A along $+z$ direction.

(i) Set $\vec{B} = \vec{e}_\phi \, B_\phi(\rho)$ (justify this briefly), use Ampere's Law (choose and draw appropriate closed loop) to deduce that the magnetic field at a field point outside the given thin conducting wire is

$$\vec{B} = \vec{e}_\phi \frac{\mu_0 \, I}{2\,\pi\,\rho} \quad \cdots\cdots \quad (1)$$

where ρ is the distance from z-axis and $\vec{e}_\phi$ is one of the unit vectors in cylindrical coordinate system. **(1+1+4 marks)**

(ii) Given a vector potential $\vec{A} = \vec{e}_z \left(-\frac{\mu_0 \, I}{2\,\pi} \ln(\rho) \right)$, evaluate $\vec{\nabla} \times \vec{A}$ and show that it yields the same result as the right hand side expression of eq.(1), i.e., the given vector potential can be a correct vector potential for this problem. **(5 marks)**

(b) Two very long thin conducting wires parallel to z-axis and lying on the $y = 0$ plane, i.e, $x - z$ *plane*, one situated at $x = -b$ and carries a static current I A along -z direction and the other situated at $x = +b$ and carries a static current I A along +z direction as shown in the Figure.1 (on $y = 0$ plane) and Figure.2 (on z = constant plane) below.

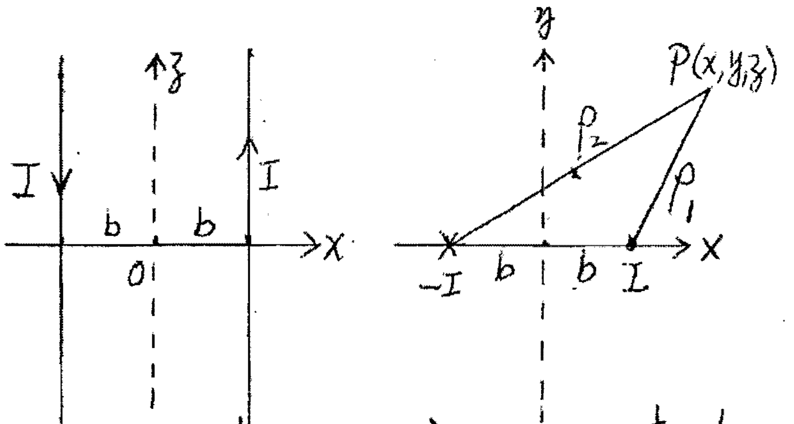


Fig.1 y=0 plane Fig.2 z=const. plane

(i) Utilize the result in (a)(ii), apply the superposition principle to deduce that the vector potential at point $P:(x,y,z)$ due to the two parallel conducting wires is

$$\vec{A}(x,y,z) = \vec{e}_z \frac{\mu_0 \, I}{4\,\pi} \ln\left(\frac{(x+b)^2 + y^2}{(x-b)^2 + y^2} \right) \quad \cdots\cdots \quad (2)$$

(5 marks)

(ii) Use $\vec{B} = \vec{\nabla} \times \vec{A}$ and $\vec{A}$ from eq.(2), and find the magnetic field $\vec{B}$ at point $P:(x,y,z)$ and show that

$$\vec{B} = -\vec{e}_x \frac{\mu_0 \, I \, (2\,b\,x\,y)}{\pi \left((x+b)^2 + y^2\right)\left((x-b)^2 + y^2\right)} - \vec{e}_y \frac{\mu_0 \, I \, b \left(y^2 - x^2 + b^2\right)}{\pi \left((x+b)^2 + y^2\right)\left((x-b)^2 + y^2\right)}$$

(9 marks)

Question three

(a) In a conductive region, based on Drude's model the force on a conduction electron by $\vec{E}$ is $-e\,\vec{E}$. The retardation force by the ion lattice of the conductor is $-\dfrac{m_e\,\vec{v}_d}{\tau_c}$.

Therefore the equation of motion for an average conduction electron in the conductor is

$$m_e \frac{d\vec{v}_d}{dt} = -e\,\vec{E} - \frac{m_e\,\vec{v}_d}{\tau_c} \quad \cdots\cdots \quad (1)\ .$$

(i) What are $\vec{v}_d$ and τ_c in the above equation? **(2 marks)**

(ii) The current density in the metal can be expressed as $\vec{J} = \rho_v\,\vec{v}_d = -n\,e\,\vec{v}_d$.

(A) What does n represent in the above expression. **(1 mark)**

(B) Substitute the above expression into eq.(1) and deduce that

$$\tau_c \frac{d\vec{J}}{dt} + \vec{J} = \frac{n\,e^2\,\tau_c}{m_e}\vec{E} \quad \cdots\cdots \quad (2)$$ **(3 marks)**

(C) Use the point form of Ohm's law $\vec{J} = \sigma\,\vec{E}$ and deduce from eq.(2) that the d.c. conductivity of the pure metal $\sigma_{d.c.}$ is

$$\sigma_{d.c.} = \frac{n\,e^2\,\tau_c}{m_e} \quad \cdots\cdots \quad (3)\ .$$ **(1 mark)**

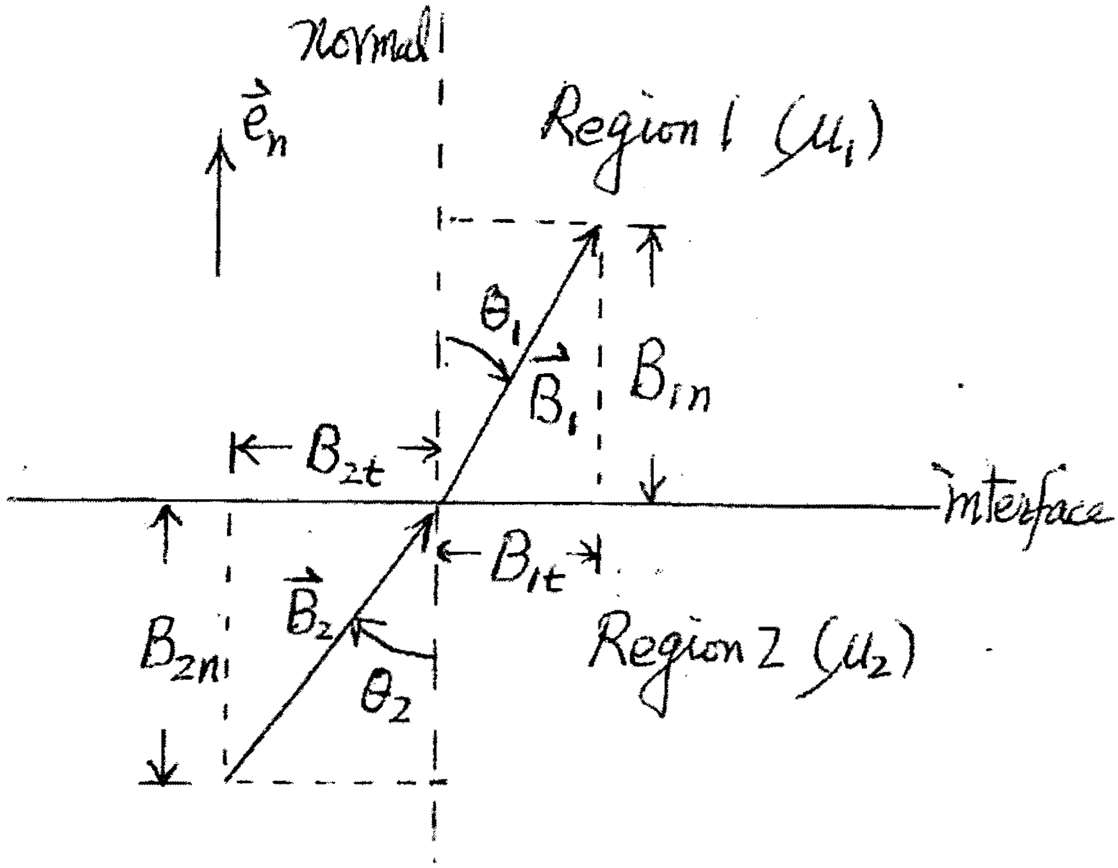
(D) If $\vec{E}$ is assumed sinusoidal with angular frequency ω , i.e., $\vec{E}$ & $\vec{J}$ in eq.(2) can be replaced by $\hat{\vec{E}}\,e^{i\omega t}$ & $\hat{\vec{J}}\,e^{i\omega t}$ respectively , deduce that

$$\hat{\sigma}_{a.c.} = \frac{n\,e^2\,\tau_c}{m_e + i\,m_e\,\omega\,\tau_c} \quad \cdots\cdots \quad (4)\ .$$ **(4 mark)**

(Hint : $\hat{\vec{J}} = \hat{\sigma}_{a.c.}\,\hat{\vec{E}}$ & $\hat{\sigma}_{a.c.} = |\hat{\sigma}_{a.c.}|$)

(iii) If a certain pure metal having an atomic density of 5.8×10^{28} *atom/m³* at room temperature and two outer orbit electrons available for conduction, find the value of τ_c if its measured d.c. conductivity is $\sigma = 3.6 \times 10^{7}\ \dfrac{1}{m\,\Omega}$. **(4 marks)**

(b) An interface separating two isotropic materials bave permeabilities μ_1 & μ_2 . $\vec{B}_1$ & $\vec{B}_2$ are the magnetic fields at points on either side of the interface infinitely close to each other and θ_1 & θ_2 are their respective angles made with the normal as shown in the diagram below.



(i) Use integral magnetic Gauss law , choose and draw a proper Gaussian closed surface (pillbox in shape across the interface) to deduce that the normal component of $\vec{B}$ is continuous at the interface , i.e., $B_{1n} = B_{2n}$. **(1+4 marks)**

(ii) Together with the tangential component of $\vec{H}$ is continuous at the interface , i.e., $H_{1t} = H_{2t}$, deduce the following refraction relation for $\vec{B}$ as

$$\tan(\theta_2) = \frac{\mu_2}{\mu_1}\tan(\theta_1) \ .$$ **(5 marks)**

Question four

(a) Starting with the following time harmonic Maxwell's equations for a material region represented by parameters of μ , ε & σ as

$$\begin{cases} \vec{\nabla} \bullet \hat{\vec{E}}(\vec{r}) = 0 & \cdots\cdots \quad (1) \\ \vec{\nabla} \bullet \hat{\vec{H}}(\vec{r}) = 0 & \cdots\cdots \quad (2) \\ \vec{\nabla} \times \hat{\vec{E}}(\vec{r}) = -i\,\omega\,\mu\,\hat{\vec{H}}(\vec{r}) & \cdots\cdots \quad (3) \\ \vec{\nabla} \times \hat{\vec{H}}(\vec{r}) = (\sigma + i\,\omega\,\varepsilon)\,\hat{\vec{E}}(\vec{r}) & \cdots\cdots \quad (4) \end{cases}$$

and further assuming that $\hat{\vec{E}}(\vec{r})$ & $\hat{\vec{H}}(\vec{r})$ are functions of z only , i.e.,

$$\hat{\vec{E}}(\vec{r}) = \vec{a}_x\ \hat{E}_x(z) + \vec{a}_y\ \hat{E}_y(z) + \vec{a}_z\ \hat{E}_z(z) \qquad \cdots\cdots \quad (5)$$

$$\hat{\vec{H}}(\vec{r}) = \vec{a}_x\ \hat{H}_x(z) + \vec{a}_y\ \hat{H}_y(z) + \vec{a}_z\ \hat{H}_z(z) \qquad \cdots\cdots \quad (6)$$

deduce that

(i) $\hat{E}_z(z) = 0 = \hat{H}_z(z)$. **(2 marks)**

(ii) $\dfrac{d^2\,\hat{E}_x(z)}{d\,z^2} = -\,\hat{\gamma}^2\ \hat{E}_x(z)$ *where* $\hat{\gamma} = \sqrt{(\omega^2\ \mu\,\varepsilon - i\,\omega\,\mu\,\sigma)}$. **(9 marks)**

(b) An uniform plane wave travelling along $+z$ direction with the field components of (E_x , H_y) has a complex electric field amplitude of $100\ e^{i\,60^0}$ *V/m* and propagates at $f = 10^6$ *Hz* in a material region having the parameters of $\mu = \mu_0$, $\varepsilon = 9\,\varepsilon_0$ & $\dfrac{\sigma}{\omega\,\varepsilon} = 1$.

(i) Find the values of the propagation constant $\hat{\gamma}$ $(= \alpha + i\,\beta)$ and the intrinsic wave impedance $\hat{\eta}$ for this wave . **(4 marks)**

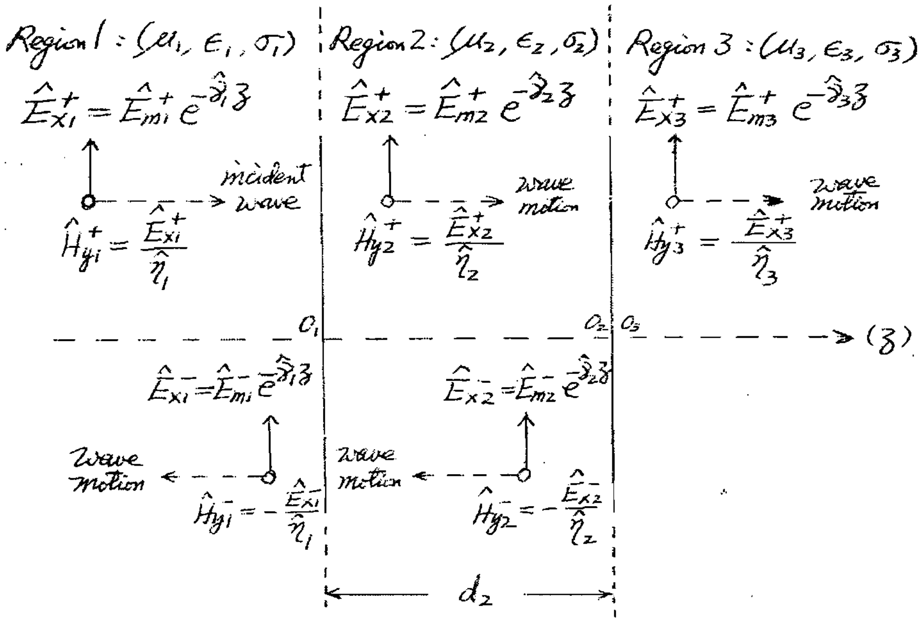
(ii) Express the electric and magnetic fields in both their complex and real-time forms, with the numerical values of (b)(i) inserted . **(4 marks)**

(iii) Use the real time expression of $\vec{E}$ & $\vec{H}$ of the given travelling wave to evaluate the poynting vector $\vec{P}$ $(\equiv \vec{E} \times \vec{H})$ of the wave . **(2 marks)**

(iv) Find the values of the penetration depth, wave length and phase velocity of the given wave. Also find the index of refraction of the given material at this given propagation frequency. **(4 marks)**

Question five

An uniform plane wave $\left(\hat{E}_{x1}^{+}\ ,\ \hat{H}_{y1}^{+}\right)$, operating at a frequency f , is normally incident upon a layer of d_2 thickness, and emerges to region 3 as shown below :



0_1 , 0_2 & 0_3 are the respective origins for region 1 , 2 & 3 chosen at the first and second interface .

(a) Define for the i^{th} *region* $(i = 1,2,3)$ the reflection coefficient $\hat{\Gamma}_i(z)$ and the total wave impedance $\hat{Z}_i(z)$ and deduce the following :

$$\begin{cases} \hat{Z}_i(z) = \hat{\eta}_i \dfrac{1+\hat{\Gamma}_i(z)}{1-\hat{\Gamma}_i(z)} \\ \hat{\Gamma}_i(z') = \hat{\Gamma}_i(z)\, e^{2\hat{\gamma}_i (z'-z)} \quad \text{where } z' \ \& \ z \text{ are two positions in } i^{th} \text{ region} \end{cases}$$

(5 + 4 marks)

(b) If $f = 10^5\ Hz$ & $d_2 = \dfrac{\lambda_2}{4}$, region 1 & 3 are air regions and region 2 is a lossless region with parameters $\mu_2 = \mu_0$, $\varepsilon_1 = 9\,\varepsilon_0$ & $\dfrac{\sigma}{\omega\,\varepsilon} = 0$

(i) find the values of β_1 , β_2 , β_3 , λ_2 & $\hat{\eta}_2$, (note : $\hat{\eta}_1 = \hat{\eta}_3 = 120\,\pi\ \Omega$ and $\alpha_1 = \alpha_2 = \alpha_3 = 0$) **(4 marks)**

(ii) starting with $\hat{\Gamma}_3(z) = 0$ for region 3 , and using continuous $\hat{Z}$ at the interface as well as the equations in (a) , find the values of

$\hat{Z}_3(0)$, $\hat{Z}_2(0)$, $\hat{\Gamma}_2(0)$, $\hat{\Gamma}_2(-10\ cm)$, $\hat{Z}_2(-10\ cm)$, $\hat{Z}_1(0)$ & $\hat{\Gamma}_1(0)$

(10 marks)

(iii) find the value of $\hat{E}_{m1}^{-}$ *if given* $\hat{E}_{m1}^{+} = 100\, e^{i\,50^0}$ *V/m.* **(2 marks)**

Useful informations

$$e = 1.6 \times 10^{-19}\ C$$

$$m_e = 9.1 \times 10^{-31}\ kg$$

$$\mu_0 = 4\,\pi \times 10^{-7}\ \frac{H}{m}$$

$$\varepsilon_0 = 8.85 \times 10^{-12}\ \frac{F}{m}$$

$\hat{\gamma} = \alpha + i\,\beta$ where

$$\alpha = \frac{\omega\sqrt{\mu\,\varepsilon}}{\sqrt{2}}\sqrt{\sqrt{1+\left(\frac{\sigma}{\omega\,\varepsilon}\right)^2}-1}$$

$$\beta = \frac{\omega\sqrt{\mu\,\varepsilon}}{\sqrt{2}}\sqrt{\sqrt{1+\left(\frac{\sigma}{\omega\,\varepsilon}\right)^2}+1}$$

$$\frac{1}{\sqrt{\mu_0\,\varepsilon_0}} = 3 \times 10^{8}\ \frac{m}{s}$$

$$\hat{\eta} = \frac{\sqrt{\frac{\mu}{\varepsilon}}}{\sqrt[4]{1+\left(\frac{\sigma}{\omega\,\varepsilon}\right)^2}}\, e^{i\frac{1}{2}\tan^{-1}\left(\frac{\sigma}{\omega\,\varepsilon}\right)}$$

$$\eta_0 = 120\,\pi\ \Omega = 377\ \Omega$$

$$\beta_0 = \omega\sqrt{\mu_0\,\varepsilon_0}$$

$$\oint\!\!\oint_S \vec{E} \bullet d\vec{s} = \frac{1}{\varepsilon}\iiint_V \rho_v\, dv$$

$$\oint\!\!\oint_S \vec{B} \bullet d\vec{s} \equiv 0$$

$$\oint_L \vec{E} \bullet d\vec{l} = -\frac{\partial}{\partial t}\left(\iint_S \vec{B} \bullet d\vec{s}\right)$$

$$\oint_L \vec{B} \bullet d\vec{l} = \mu \iint_S \vec{J} \bullet d\vec{s} + \mu\varepsilon\frac{\partial}{\partial t}\left(\iint_S \vec{E} \bullet d\vec{s}\right)$$

$$\vec{\nabla} \bullet \vec{E} = \frac{\rho_v}{\varepsilon}$$

$$\vec{\nabla} \bullet \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \times \vec{B} = \mu \vec{J} + \mu \varepsilon \frac{\partial \vec{E}}{\partial t}$$

$$\vec{J} = \sigma \vec{E}$$

$$\oiint_S \vec{F} \bullet d\vec{s} \equiv \oiiint_V \left(\vec{\nabla} \bullet \vec{F}\right) d v \qquad \textit{divergence theorem}$$

$$\oint_L \vec{F} \bullet d\vec{l} \equiv \iint_S \left(\vec{\nabla} \times \vec{F}\right) \bullet d\vec{s} \qquad \textit{Stokes' theorem}$$

$$\vec{\nabla} \bullet \left(\vec{\nabla} \times \vec{F}\right) \equiv 0$$

$$\vec{\nabla} \times \left(\vec{\nabla} f\right) \equiv 0$$

$$\vec{\nabla} \times \left(\vec{\nabla} \times \vec{F}\right) \equiv \vec{\nabla}\left(\vec{\nabla} \bullet \vec{F}\right) - \nabla^2 \vec{F}$$

$$\vec{\nabla} f = \vec{e}_x \frac{\partial f}{\partial x} + \vec{e}_y \frac{\partial f}{\partial y} + \vec{e}_z \frac{\partial f}{\partial z} = \vec{e}_\rho \frac{\partial f}{\partial \rho} + \vec{e}_\phi \frac{1}{\rho} \frac{\partial f}{\partial \phi} + \vec{e}_z \frac{\partial f}{\partial z}$$

$$= \vec{e}_r \frac{\partial f}{\partial r} + \vec{e}_\theta \frac{1}{r} \frac{\partial f}{\partial \theta} + \vec{e}_\phi \frac{1}{r \sin(\theta)} \frac{\partial f}{\partial \phi}$$

$$\vec{\nabla} \bullet \vec{F} = \frac{\partial(F_x)}{\partial x} + \frac{\partial(F_y)}{\partial y} + \frac{\partial(F_z)}{\partial z} = \frac{1}{\rho} \frac{\partial(F_\rho \, \rho)}{\partial \rho} + \frac{1}{\rho} \frac{\partial(F_\phi)}{\partial \phi} + \frac{\partial(F_z)}{\partial z}$$

$$= \frac{1}{r^2} \frac{\partial(F_r \, r^2)}{\partial r} + \frac{1}{r \sin(\theta)} \frac{\partial(F_\theta \sin(\theta))}{\partial \theta} + \frac{1}{r \sin(\theta)} \frac{\partial(F_\phi)}{\partial \phi}$$

$$\vec{\nabla} \times \vec{F} = \vec{e}_x \left(\frac{\partial(F_z)}{\partial y} - \frac{\partial(F_y)}{\partial z}\right) + \vec{e}_y \left(\frac{\partial(F_x)}{\partial z} - \frac{\partial(F_z)}{\partial x}\right) + \vec{e}_z \left(\frac{\partial(F_y)}{\partial x} - \frac{\partial(F_x)}{\partial y}\right)$$

$$= \frac{\vec{e}_\rho}{\rho} \left(\frac{\partial(F_z)}{\partial \phi} - \frac{\partial(F_\phi \, \rho)}{\partial z}\right) + \vec{e}_\phi \left(\frac{\partial(F_\rho)}{\partial z} - \frac{\partial(F_z)}{\partial \rho}\right) + \frac{\vec{e}_z}{\rho} \left(\frac{\partial(F_\phi \, \rho)}{\partial \rho} - \frac{\partial(F_\rho)}{\partial \phi}\right)$$

$$= \frac{\vec{e}_r}{r^2 \sin(\theta)} \left(\frac{\partial(F_\phi \, r \sin(\theta))}{\partial \theta} - \frac{\partial(F_\theta \, r)}{\partial \phi}\right) + \frac{\vec{e}_\theta}{r \sin(\theta)} \left(\frac{\partial(F_r)}{\partial \phi} - \frac{\partial(F_\phi \, r \sin(\theta))}{\partial r}\right) + \frac{\vec{e}_\phi}{r} \left(\frac{\partial(F_\theta \, r)}{\partial r} - \frac{\partial(F_r)}{\partial \theta}\right)$$

where $\vec{F} = \vec{e}_x F_x + \vec{e}_y F_y + \vec{e}_z F_z = \vec{e}_\rho F_\rho + \vec{e}_\phi F_\phi + \vec{e}_z F_z = \vec{e}_r F_r + \vec{e}_\theta F_\theta + \vec{e}_\phi F_\phi$ and

$d\vec{l} = \vec{e}_x \, dx + \vec{e}_y \, dy + \vec{e}_z \, dz = \vec{e}_\rho \, d\rho + \vec{e}_\phi \, \rho \, d\phi + \vec{e}_z \, dz = \vec{e}_r \, dr + \vec{e}_\theta \, r \, d\theta + \vec{e}_\phi \, r \sin(\theta) \, d\phi$

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial x^2} = \frac{1}{\rho} \frac{\partial}{\partial \rho}\left(\rho \frac{\partial f}{\partial \rho}\right) + \frac{1}{\rho^2} \frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2}$$

$$= \frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 \frac{\partial f}{\partial r}\right) + \frac{1}{r^2 \sin(\theta)} \frac{\partial}{\partial \theta}\left(\sin(\theta) \frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2 \sin^2(\theta)} \frac{\partial^2 f}{\partial \phi^2}$$