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UNIVERSITY OF SWAZILAND

FACULTY OF SCIENCE

DEPARTMENT OF PHYSICS

SUPPLEMENTARY EXAMINATION 2011/2012

TITLE OF PAPER	:	ELECTROMAGNETIC THEORY I
COURSE NUMBER	:	P331
TIME ALLOWED	:	THREE HOURS
INSTRUCTIONS	:	ANSWER ANY FOUR OUT OF FIVE QUESTIONS. EACH QUESTION CARRIES 25 MARKS. MARKS FOR DIFFERENT SECTIONS ARE SHOWN IN THE RIGHT-HAND MARGIN.

THIS PAPER HAS NINE PAGES, INCLUDING THIS PAGE.

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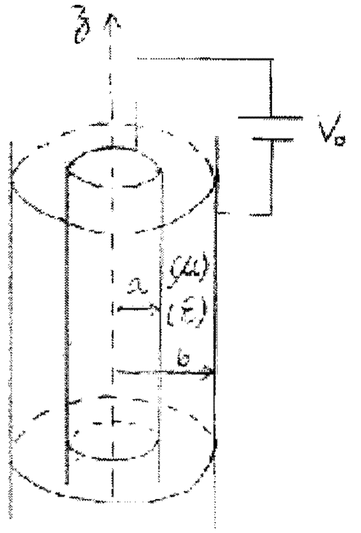
P331 Electromagnetic Theory I

Question one

A very long straight coaxial cable has its central axis aligned with z-axis , its inner and outer conducting cable's cross-sectional radius are a & b respectively and in-between the cables is a insulating layer of permittivity ε , as shown below :



(a) From $\nabla^2 f(\rho) = 0$, i.e., $\dfrac{d\left(\rho \dfrac{d\,f(\rho)}{d\rho}\right)}{d\rho} = 0$, with boundary conditions of $f(\rho = b) = 0$ and $f(\rho = a) = V_0$, find the specific solution of $f(\rho)$ and show that $f(\rho) = -\dfrac{V_0}{\ln\left(\dfrac{b}{a}\right)}(\ln(\rho) - \ln(b))$. **(10 marks)**

(b) Use $\vec{E} = -\vec{\nabla} f$ to find the electric field $\vec{E}$ in the insulating layer , i.e., $a \le \rho \le b$. **(4 marks)**

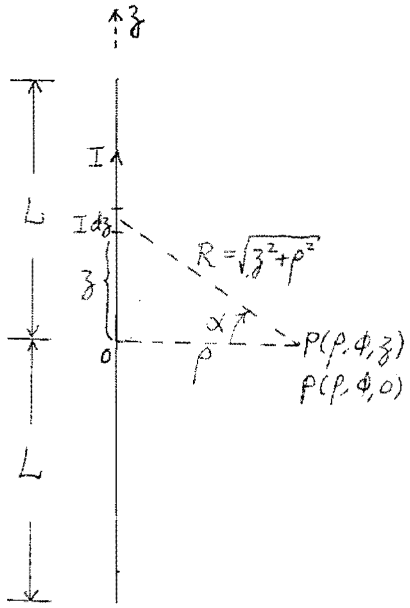
(c) From $\rho_s = \varepsilon\, E_n$ find the electric surface charge density ρ_s on $\rho = a$ and $\rho = b$ surfaces respectively . Then find the total charges deposited on both conducting cable surfaces (i.e., on $\rho = a$ & $\rho = b$ surfaces) of one meter length and show that they are equal and opposite. **(9 marks)**

(d) Find the distributive capacitance of this coaxial cable system in terms of a , b & ε . **(2 marks)**

Question two

(a) For the time-independent (or static) case, setting $\vec{B} = \vec{\nabla} \times \vec{A}$ and using Coulomb's gauge, i.e., $\vec{\nabla} \bullet \vec{A} = 0$, to deduce the following Poisson's equations for $\vec{A}$ from Maxwell's equations as $\nabla^2 \vec{A} = -\mu \vec{J}$. **(5 marks)**

(b) A straight thin conducting wire of length 2L carries a total current I along +z direction as shown in the diagram below :



(i) find the z-component of vector potential $\vec{A}$, i.e., A_z, due to the above given current at a point $P:(\rho,\phi,0)$ by evaluating the following integral

$$A_z = \int_{z=-L}^{L} \frac{\mu_0\, I\, dz}{4\pi R} = 2\int_{z=0}^{L} \frac{\mu_0\, I\, dz}{4\pi\sqrt{z^2+\rho^2}}$$ and show that

$$A_z = \frac{\mu_0\, I}{2\pi}\left(\ln\left(\frac{\sqrt{\rho^2+L^2}+L}{\rho}\right)\right)$$ **(10 marks)**

(Hint : $d\tan(\alpha) = \sec^2(\alpha)\,d\alpha$ & $1+\tan^2(\alpha) = \sec^2(\alpha)$; $\int \sec(\alpha)\,d\alpha = \ln(\sec(\alpha)+\tan(\alpha))$)

(ii) From $\vec{B} = \vec{\nabla} \times \vec{A}$ and knowing $A_\rho = 0 = A_\phi$, find the magnetic field $\vec{B}$ at the point P due to the above given current. In the case of $L \gg \rho$, show that magnetic field $\vec{B}$ obtained here can be simplified to $\vec{B} = \vec{a}_\rho \dfrac{\mu_0\, I}{2\pi\rho}$. **(10 marks)**

Question three

Starting with the following Maxwell's equations for a material region with parameters of $(\mu, \varepsilon, \sigma)$ as :

$$\begin{cases} \vec{\nabla} \bullet \vec{E}(space,t) = 0 & \cdots\cdots \quad (1) \\ \vec{\nabla} \bullet \vec{H}(space,t) = 0 & \cdots\cdots \quad (2) \\ \vec{\nabla} \times \vec{E}(space,t) = -\mu \dfrac{\partial \vec{H}(space,t)}{\partial t} & \cdots\cdots \quad (3) \\ \vec{\nabla} \times \vec{H}(space,t) = \sigma \vec{E}(space,t) + \varepsilon \dfrac{\partial \vec{E}(space,t)}{\partial t} & \cdots\cdots \quad (4) \end{cases}$$

(a) setting $\vec{E}(space,t) = \hat{\vec{E}}(space)\, e^{i\omega t}$ & $\vec{H}(space,t) = \hat{\vec{H}}(space)\, e^{i\omega t}$, deduce the following time-harmonic Maxwell's equations :

$$\begin{cases} \vec{\nabla} \bullet \hat{\vec{E}}(space) = 0 & \cdots\cdots \quad (5) \\ \vec{\nabla} \bullet \hat{\vec{H}}(space) = 0 & \cdots\cdots \quad (6) \\ \vec{\nabla} \times \hat{\vec{E}}(space) = -i\omega\mu\, \hat{\vec{H}}(space) & \cdots\cdots \quad (7) \\ \vec{\nabla} \times \hat{\vec{H}}(space) = (\sigma + i\omega\varepsilon)\, \hat{\vec{E}}(space) & \cdots\cdots \quad (8) \end{cases}$$

(3 marks)

(b) consider the space dependence of $\hat{\vec{E}}$ & $\hat{\vec{H}}$ are on z only , i.e.,

$$\hat{\vec{E}}(space) = \hat{\vec{E}}(z) = \vec{e}_x\, \hat{E}_x(z) + \vec{e}_y\, \hat{E}_y(z) + \vec{e}_z\, \hat{E}_z(z)$$

$$\hat{\vec{H}}(space) = \hat{\vec{H}}(z) = \vec{e}_x\, \hat{H}_x(z) + \vec{e}_y\, \hat{H}_y(z) + \vec{e}_z\, \hat{H}_z(z)$$

deduce the following equations :

$$\hat{E}_z(z) = constant \quad \cdots\cdots \quad (9)$$

$$\hat{H}_z(z) = constant \quad \cdots\cdots \quad (10)$$

$$\frac{d\hat{E}_y(z)}{dz} = i\omega\mu\, \hat{H}_x(z) \quad \cdots\cdots \quad (11)$$

$$\frac{d\hat{E}_x(z)}{dz} = -i\omega\mu\, \hat{H}_y(z) \quad \cdots\cdots \quad (12)$$

$$\hat{H}_z(z) = 0 \quad \cdots\cdots \quad (13)$$

$$\frac{d\hat{H}_y(z)}{dz} = -(\sigma + i\omega\varepsilon)\, \hat{E}_x(z) \quad \cdots\cdots \quad (14)$$

$$\frac{d\hat{H}_x(z)}{dz} = (\sigma + i\omega\varepsilon)\, \hat{E}_y(z) \quad \cdots\cdots \quad (15)$$

$$\hat{E}_z(z) = 0 \quad \cdots\cdots \quad (16)$$

(10 marks)

Question three (continued)

(c) From equations in (b) deduce the following wave equation for $\hat{E}_y(z)$ as

$$\frac{d^2\,\hat{E}_y(z)}{d\,z^2} = i\,\omega\,\mu(\sigma + i\,\omega\,\varepsilon)\,\hat{E}_y(z) \quad \cdots\cdots \quad (17)$$

then by direct substitution show that

$\hat{E}_y(z) = \hat{E}_m^+\, e^{-\hat{\gamma}\, z} + \hat{E}_m^-\, e^{+\hat{\gamma}\, z} \quad \cdots (18)$ *where* $\hat{\gamma} = \sqrt{i\,\omega\,\mu(\sigma + i\,\omega\,\varepsilon)}$ *and*

$\hat{E}_m^+$ & $\hat{E}_m^-$ *are complex constants (represents + & − z complex amplitudes)*

is solution to eq(17). **(7 marks)**

(d) Substitute eq(18) into one of the equations in (b) and deduce that its wave partner $\hat{H}_x(z)$ is

$\hat{H}_x(z) = \hat{H}_m^+\, e^{-\hat{\gamma}\, z} + \hat{H}_m^-\, e^{+\hat{\gamma}\, z} \quad \cdots (19)$ *where*

$$-\frac{\hat{E}_m^+}{\hat{H}_m^+} = +\frac{\hat{E}_m^-}{\hat{H}_m^-} = \sqrt{\frac{i\,\omega\,\mu}{\sigma + i\,\omega\,\varepsilon}} \xrightarrow{\text{set as}} \hat{\eta}$$

(5 marks)

Question four

(a) Given the following data for Cesiummetal Cs as :
Atomic weight = 132.9 gm / gm-mole
Density = 1873 kg / m^3

Conductivity $\sigma = 4.9 \times 10^7 \ \dfrac{1}{\Omega \, m}$ at room temperature

(i) Find the number of conduction electrons per meter cube, i.e., number density n , for metal Cs if each Cs atom has one conduction electron. **(4 marks)**
(Hint : one gm – mole pure metal contains 6.022×10^{23} *atoms*)

(ii) Find the value of the mean free time t_f for metal Cs at room temperature.
(Hint : $t_f = \dfrac{2 \, m_e \, \sigma}{n \, e^2}$) **(4 marks)**

(iii) If the applied constant electric field if $\vec{E} = \vec{a}_y \, 100 \quad V/m$, find the saturated drifting velocity $\vec{v}_d$ of the average conducting electron of metal Cs at room temperarure. **(4 marks)**
(Hint : $\vec{v}_d = -\dfrac{e \, t_f}{2 \, m_e} \vec{E}$)

(b) An uniform plane wave travelling along + z direction with the field components of (E_x , H_y) has a complex electric field amplitude of $100 \, e^{i \, 50^0} \ \dfrac{V}{m}$ and propagates at $f = 5 \times 10^6 \ Hz$ in a material region having the parameters of $\mu = \mu_0$, $\varepsilon = 4 \, \varepsilon_0$ & $\dfrac{\sigma}{\omega \, \varepsilon} = 1$,

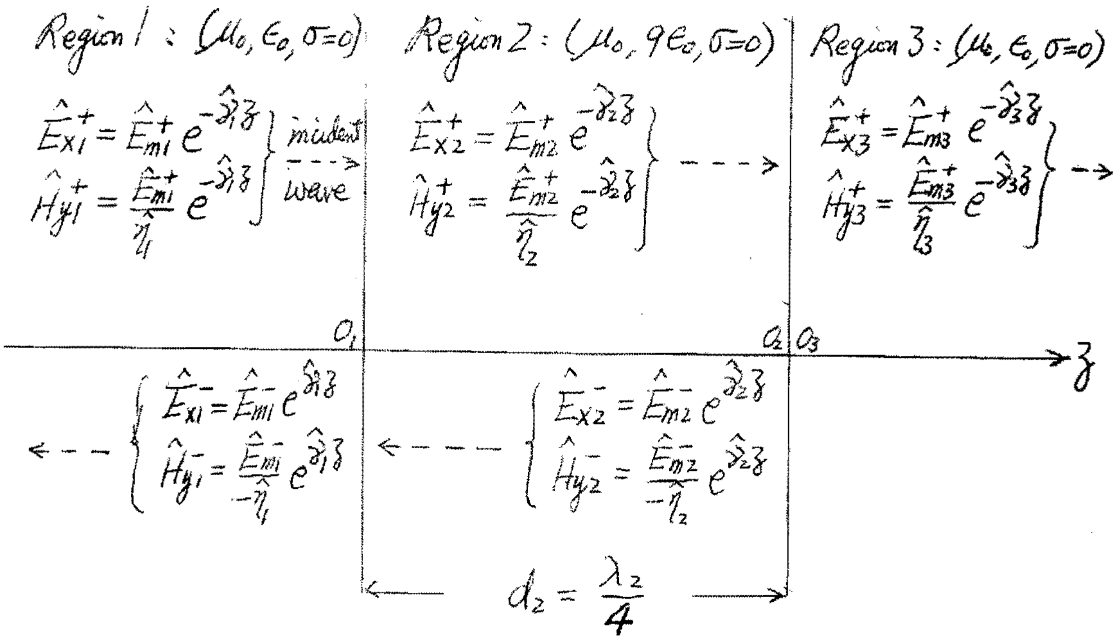
(i) Find the values of the propagation constant $\hat{\gamma} \ (= \alpha + i \, \beta)$ and the intrinsic wave impedance $\hat{\eta}$ for this wave . **(5 marks)**

(ii) express the electric and magnetic fields in both their complex and real-time forms, with the numerical values of (b)(i) inserted , **(5 marks)**

(iii) find the values of the penetration depth , wave length and phase velocity of the given wave. **(3 marks)**

Question five

An uniform plane wave $\left(\hat{E}_{x1}^{+}, \hat{H}_{y1}^{+}\right)$, operates at $f = 10^8 \ Hz$, is normally incident upon a lossless layer of $d_2 \left(= \frac{\lambda_2}{4}\right)$ thickness with parameters of $\left(\mu = \mu_0, \varepsilon = 9\,\varepsilon_0\right)$ as shown below :



0_1 , 0_2 & 0_3 are the respective origins for region 1 , 2 & 3 chosen at the first and second interface . (Both region 1 and region 3 are air region.)

(a) Find the values of $\hat{\gamma}_1$, $\hat{\gamma}_2$, $\hat{\gamma}_3$, $\hat{\eta}_2$ & λ_2 , (note : $\hat{\eta}_1 = \hat{\eta}_3 = 120\,\pi\ \Omega$ and $\alpha_1 = \alpha_2 = \alpha_3 = 0$) **(4 marks)**

(b) Starting with $\hat{\Gamma}_3(z) = 0$ for the rightmost region , i.e., region 3 , and using continuous $\hat{Z}$ at the interface as well as the equations in (a) , find the values of $\hat{Z}_3(0)$, $\hat{Z}_2(0)$, $\hat{\Gamma}_2(0)$, $\hat{\Gamma}_2(-d_2)$, $\hat{Z}_2(-d_2)$, $\hat{Z}_1(0)$ & $\hat{\Gamma}_1(0)$ **(10 marks)**

(c) Find the value of $\hat{E}_{m1}^{-}$, $\hat{E}_{m2}^{+}$, $\hat{E}_{m2}^{-}$ & $\hat{E}_{m3}^{+}$ if given $\hat{E}_{m1}^{+} = 50\, e^{j\,40^0}\ \frac{V}{m}$. **(11 marks)**

Useful informations

$$e = 1.6 \times 10^{-19}\ C$$

$$m_e = 9.1 \times 10^{-31}\ kg$$

$$\mu_0 = 4\pi \times 10^{-7}\ \frac{H}{m}$$

$$\varepsilon_0 = 8.85 \times 10^{-12}\ \frac{F}{m}$$

$$\alpha = \frac{\omega\sqrt{\mu\varepsilon}}{\sqrt{2}}\sqrt{\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}-1}$$

$$\beta = \frac{\omega\sqrt{\mu\varepsilon}}{\sqrt{2}}\sqrt{\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}+1}$$

$$\frac{1}{\sqrt{\mu_0\varepsilon_0}} = 3 \times 10^8\ \frac{m}{s}$$

$$\hat{\eta} = \frac{\sqrt{\frac{\mu}{\varepsilon}}}{\sqrt[4]{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}}\, e^{i\frac{1}{2}\tan^{-1}\left(\frac{\sigma}{\omega\varepsilon}\right)}$$

$$\eta_0 = 120\pi\ \Omega = 377\ \Omega$$

$$\beta_0 = \omega\sqrt{\mu_0\varepsilon_0}$$

$$\oiint_S \vec{E} \bullet d\vec{s} = \frac{1}{\varepsilon}\iiint_V \rho_v\, dv$$

$$\oiint_S \vec{B} \bullet d\vec{s} \equiv 0$$

$$\oint_L \vec{E} \bullet d\vec{l} = -\frac{\partial}{\partial t}\left(\iint_S \vec{B} \bullet d\vec{s}\right)$$

$$\oint_L \vec{B} \bullet d\vec{l} = \mu\iint_S \vec{J} \bullet d\vec{s} + \mu\varepsilon\frac{\partial}{\partial t}\left(\iint_S \vec{E} \bullet d\vec{s}\right)$$

$$\vec{\nabla} \bullet \vec{E} = \frac{\rho_v}{\varepsilon}$$

$$\vec{\nabla} \bullet \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

$$\vec{\nabla} \times \vec{B} = \mu\vec{J} + \mu\varepsilon\frac{\partial\vec{E}}{\partial t}$$

$$\vec{J} = \sigma\vec{E}$$

$$\oiint_S \vec{F} \bullet d\vec{s} \equiv \oiiint_V \left(\vec{\nabla} \bullet \vec{F}\right) d v \qquad \textit{divergence theorem}$$

$$\oint_L \vec{F} \bullet d\vec{l} \equiv \iint_S \left(\vec{\nabla} \times \vec{F}\right) \bullet d\vec{s} \qquad \textit{Stokes' theorem}$$

$$\vec{\nabla} \bullet \left(\vec{\nabla} \times \vec{F}\right) \equiv 0$$

$$\vec{\nabla} \times \left(\vec{\nabla} f\right) \equiv 0$$

$$\vec{\nabla} \times \left(\vec{\nabla} \times \vec{F}\right) \equiv \vec{\nabla}\left(\vec{\nabla} \bullet \vec{F}\right) - \nabla^2 \vec{F}$$

$$\vec{\nabla} f = \vec{e}_x \frac{\partial f}{\partial x} + \vec{e}_y \frac{\partial f}{\partial y} + \vec{e}_z \frac{\partial f}{\partial z} = \vec{e}_\rho \frac{\partial f}{\partial \rho} + \vec{e}_\phi \frac{1}{\rho}\frac{\partial f}{\partial \phi} + \vec{e}_z \frac{\partial f}{\partial z}$$

$$= \vec{e}_r \frac{\partial f}{\partial r} + \vec{e}_\theta \frac{1}{r}\frac{\partial f}{\partial \theta} + \vec{e}_\phi \frac{1}{r\sin(\theta)}\frac{\partial f}{\partial \phi}$$

$$\vec{\nabla} \bullet \vec{F} = \frac{\partial(F_x)}{\partial x} + \frac{\partial(F_y)}{\partial y} + \frac{\partial(F_z)}{\partial z} = \frac{1}{\rho}\frac{\partial(F_\rho\,\rho)}{\partial \rho} + \frac{1}{\rho}\frac{\partial(F_\phi)}{\partial \phi} + \frac{\partial(F_z)}{\partial z}$$

$$= \frac{1}{r^2}\frac{\partial(F_r\, r^2)}{\partial r} + \frac{1}{r\sin(\theta)}\frac{\partial(F_\theta \sin(\theta))}{\partial \theta} + \frac{1}{r\sin(\theta)}\frac{\partial(F_\phi)}{\partial \phi}$$

$$\vec{\nabla} \times \vec{F} = \vec{e}_x \left(\frac{\partial(F_z)}{\partial y} - \frac{\partial(F_y)}{\partial z}\right) + \vec{e}_y \left(\frac{\partial(F_x)}{\partial z} - \frac{\partial(F_z)}{\partial x}\right) + \vec{e}_z \left(\frac{\partial(F_y)}{\partial x} - \frac{\partial(F_x)}{\partial y}\right)$$

$$= \frac{\vec{e}_\rho}{\rho}\left(\frac{\partial(F_z)}{\partial \phi} - \frac{\partial(F_\phi\,\rho)}{\partial z}\right) + \vec{e}_\phi \left(\frac{\partial(F_\rho)}{\partial z} - \frac{\partial(F_z)}{\partial \rho}\right) + \frac{\vec{e}_z}{\rho}\left(\frac{\partial(F_\phi\,\rho)}{\partial \rho} - \frac{\partial(F_\rho)}{\partial \phi}\right)$$

$$= \frac{\vec{e}_r}{r^2\sin(\theta)}\left(\frac{\partial(F_\phi\, r\sin(\theta))}{\partial \theta} - \frac{\partial(F_\theta\, r)}{\partial \phi}\right) + \frac{\vec{e}_\theta}{r\sin(\theta)}\left(\frac{\partial(F_r)}{\partial \phi} - \frac{\partial(F_\phi\, r\sin(\theta))}{\partial r}\right) + \frac{\vec{e}_\phi}{r}\left(\frac{\partial(F_\theta\, r)}{\partial r} - \frac{\partial(F_r)}{\partial \theta}\right)$$

where $\vec{F} = \vec{e}_x F_x + \vec{e}_y F_y + \vec{e}_z F_z = \vec{e}_\rho F_\rho + \vec{e}_\phi F_\phi + \vec{e}_z F_z = \vec{e}_r F_r + \vec{e}_\theta F_\theta + \vec{e}_\phi F_\phi$ and

$$d\vec{l} = \vec{e}_x\, dx + \vec{e}_y\, dy + \vec{e}_z\, dz = \vec{e}_\rho\, d\rho + \vec{e}_\phi\, \rho\, d\phi + \vec{e}_z\, dz = \vec{e}_r\, dr + \vec{e}_\theta\, r\, d\theta + \vec{e}_\phi\, r\sin(\theta)\, d\phi$$

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial x^2} = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial f}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2}$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin(\theta)}\frac{\partial}{\partial \theta}\left(\sin(\theta)\frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2\sin^2(\theta)}\frac{\partial^2 f}{\partial \phi^2}$$

Drude's model $m_e \dfrac{d\vec{v}_d}{dt} = -e\vec{E} - \dfrac{2\, m_e\, \vec{v}_d}{t_f}$