UNIVERSITY OF SWAZILAND

136

FACULTY OF SCIENCE

DEPARTMENT OF PHYSICS

MAIN EXAMINATION 2011/2012

TITLE OF PAPER	:	**ELECTROMAGNETIC THEORY I**
COURSE NUMBER	:	**P331**
TIME ALLOWED	:	**THREE HOURS**
INSTRUCTIONS	:	**ANSWER ANY FOUR OUT OF FIVE QUESTIONS. EACH QUESTION CARRIES 25 MARKS.** **MARKS FOR DIFFERENT SECTIONS ARE SHOWN IN THE RIGHT-HAND MARGIN.**

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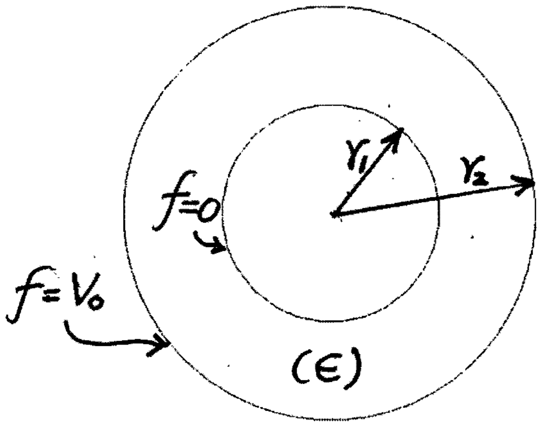
P331 Electromagnetic Theory I

Question one

A potential difference V_0 is maintained between two very thin-shelled and co-centred spherical conducting balls of radius r_1 & r_2 as depicted in the diagram below :



(a) From $\nabla^2 f(r) = 0$, i.e., $\dfrac{d\left(r^2 \dfrac{d f(r)}{dr}\right)}{dr} = 0$, with boundary conditions of $f(r = r_1) = 0$ and $f(r = r_2) = V_0$, find the specific solution of $f(r)$ and show that it is $f(r) = -\dfrac{V_0\, r_1\, r_2}{r_2 - r_1}\dfrac{1}{r} + \dfrac{V_0\, r_2}{r_2 - r_1}$. **(10 marks)**

(b) From $\vec{E} = -\vec{\nabla} f(r)$ find the electric field $\vec{E}$ established in the space between the given two conducting spheres , i.e., $r_1 \le r \le r_2$. **(3 marks)**

(c) From $\rho_s = \varepsilon\, E_n$ find the electric surface charge density ρ_s on $r = r_1$ and $r = r_2$ respectively assuming the space between the two conducting spheres is filled with the dielectric material of electric permittivity ε . Then find the total charges deposited on both $r = r_1$ & $r = r_2$ conducting surfaces and show that they are equal and opposite. **(10 marks)**

(d) Find the capacitance of this two-conductor system in terms of r_1 , r_2 & ε . **(2 marks)**

Question two

(a) A very long thin conducting wire is situated at z-axis and uniformly charged with line charge density σ_l .

(i) Use integral Coulomb's Law and choose an appropriate Gaussian surface to deduce that the electric field at a field point outside the given thin conducting wire is

$\vec{E} = \vec{e}_\rho \dfrac{\sigma_l}{2\pi\varepsilon_0\rho}$ where ρ is the distance from z-axis and

$\vec{e}_\rho$ is one of the unit vectors in cylindrical coordinate system **(7 marks)**

(ii) Use $\Phi = -\int_{P_0}^{P} \vec{E} \bullet d\vec{l}$ to find the electric potential at any point $P : (\rho, \phi, z)$ where $P_0 : (\rho_0, 0, 0)$ is the zero potential reference point and show that it is

$$\Phi = \frac{\sigma_l}{2\pi\varepsilon_0} \ln\left(\frac{\rho_0}{\rho}\right)$$ **(5 marks)**

(b) Two long thin conducting wires are parallel to the z-axis and lie in the $y = 0$ plane , i.e, in the $x - z$ *plane* . One is situated at $x = -b$ and carries $-\sigma_l$ uniform line charge density and the other is situated at $x = +b$ and carries $+\sigma_l$ uniform line charge density, as shown in the Figure.1 (on $y = 0$ plane) and Figure.2 (on $z = 0$ plane) below :

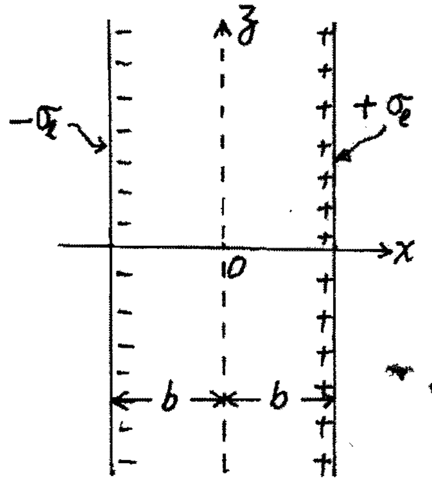


Figure 1 y=0 plane

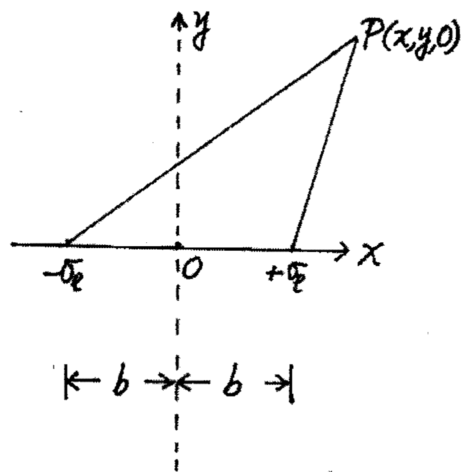


Figure 2 z=0 plane

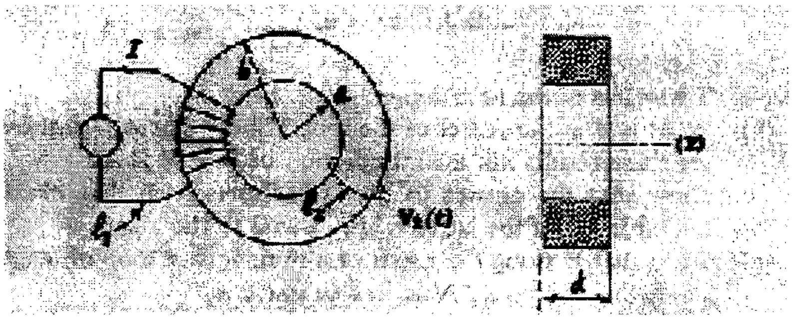
(i) Utilize the result in (a)(ii) and choose P_0 as the origin, apply the superposition principle to deduce that the electric potential at point $P : (x, y, 0)$ is

$$\Phi = \frac{\sigma_l}{4\pi\varepsilon_0} \ln\left(\frac{(x+b)^2 + y^2}{(x-b)^2 + y^2}\right)$$ **(7 marks)**

(ii) Use $\vec{E} = -\vec{\nabla}\Phi$ to find the electric field $\vec{E}$ generated by the given two conducting wires . Also find the value of $\vec{E}$ at the origin . **(4 + 2 marks)**

Question three

(a) A static current I flows in the n_1 turn toroid l_1 wired around an iron ring core of magnetic permeability μ with the rectangular cross-section area $(b-a)\times d$ as shown below



(i) Use the integral Ampere's law and choose proper closed loops to find $\vec{B}$ in terms of ρ , n_1 , μ & I within the iron core , i.e., $a \le \rho \le b$ & $0 \le z \le d$ region. **(7 marks)**

(ii) Find the total magnetic flux Ψ_m passing through the cross-section area $(b-a)\times d$ of the iron ring in counter clockwise sense, i.e., $\int_S \vec{B} \bullet d\vec{s}$ where $S: a \le \rho \le b$, $0 \le z \le d$ & $d\vec{s} = \vec{a}_\phi \, d\rho \, dz$, in terms of a, b, d, n_1, μ & I . **(6 marks)**

(iii) Find the external self-inductance L_e of the toroid wire l_1 in terms of a, b, d , μ & n_1. **(3 marks)**

(b) Placing a single turn secondary coil l_2 around the iron ring and if the toroid wire l_1 carries a sinusoidal current of $I_0 \sin(\omega t)$ instead of carrying a static current I , find the induced e.m.f. $V_2(t)$ for the single turn secondary coil l_2 in terms of a, b, d, ω, n_1 , μ & I_0 under quasi static situation . If $a = 4\ cm$, $b = 8\ cm$, $d = 2\ cm$, $n_1 = 100$, $f = 50\ Hz$, $\mu = 400\,\mu_0$ and $I_0 = 2\ A$, compute the amplitude of $V_2(t)$. **(6 + 3 marks)**

Question four

(a) (i) From the time-dependent Maxwell's equations deduce the following wave equation for $\vec{E}$ in the material region with parameters of μ , ε & σ where $\rho_v = 0$ & $\vec{J} = \sigma \vec{E}$, as

$$\nabla^2 \vec{E} = \mu \sigma \frac{\partial \vec{E}}{\partial t} + \mu \varepsilon \frac{\partial^2 \vec{E}}{\partial t^2}$$ **(6 marks)**

(ii) By direct substitution, show that $E_x = \hat{E}_m e^{\hat{\gamma} z} e^{i \omega t}$ (where $\hat{E}_m$ is any constant, ω is any frequency and $\hat{\gamma} = \sqrt{i \omega \mu \sigma - \omega^2 \mu \varepsilon}$) is a solution to the E_x part of the wave equation in (a)(i) , i.e., $\nabla^2 E_x = \mu \sigma \frac{\partial E_x}{\partial t} + \mu \varepsilon \frac{\partial^2 E_x}{\partial t^2}$.

(6 marks)

(b) An uniform plane wave traveling along the $+z$ direction with the field components $E_x(z)$ & $H_y(z)$ has a complex electric field amplitude $\hat{E}_m = 200\, e^{i\, 40^0} \frac{V}{m}$ and propagates at a frequency $f = 10^7$ Hz in a material region having the parameters $\mu = \mu_0$, $\varepsilon = 4\, \varepsilon_0$ & $\frac{\sigma}{\omega \varepsilon} = 1$.

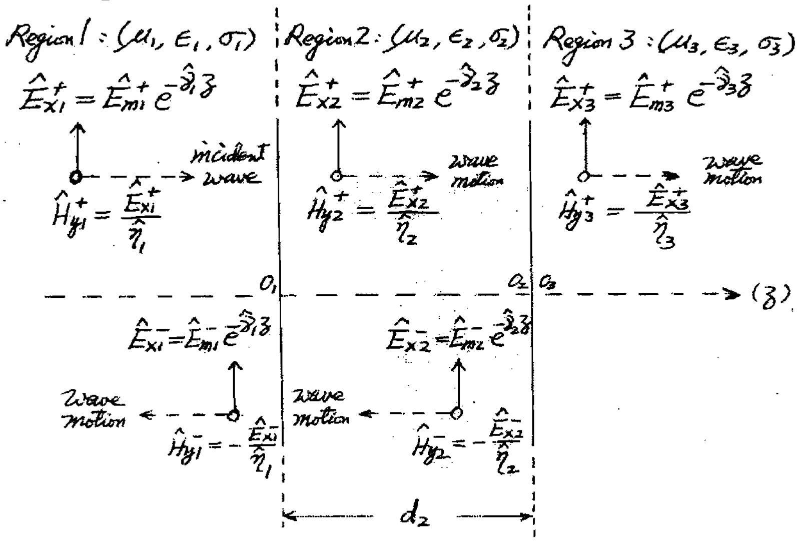
(i) Find the values of the propagation constant $\hat{\gamma}$ $(= \alpha + i\, \beta)$ and the intrinsic wave impedance $\hat{\eta}$ for this wave , **(5 marks)**

(ii) Express the electric and magnetic fields in both their complex and real-time forms , with the numerical values of (b)(i) inserted , **(5 marks)**

(iii) Find the values of the penetration depth , wavelength and phase velocity of the given wave . **(3 marks)**

Question five

An uniform plane wave $\left(\hat{E}^{+}_{x1}, \hat{H}^{+}_{y1}\right)$, operating at a frequency f, is normally incident upon a layer of d_2 thickness, and emerges to region 3 as shown below :



0_1 , 0_2 & 0_3 are the respective origins for region 1 , 2 & 3 chosen at the first and second interface .

(a) Define for the i^{th} *region* $(i = 1,2,3)$ the reflection coefficient $\hat{\Gamma}_i(z)$ and the total wave impedance $\hat{Z}_i(z)$ and deduce the following :

$$\begin{cases} \hat{Z}_i(z) = \hat{\eta}_i \dfrac{1+\hat{\Gamma}_i(z)}{1-\hat{\Gamma}_i(z)} \\ \hat{\Gamma}_i(z') = \hat{\Gamma}_i(z)\, e^{2\hat{\gamma}_i (z'-z)} \quad where\ z'\ \&\ z\ are\ two\ positions\ in\ i^{th}\ region \end{cases}$$

(5 + 4 marks)

(b) If $f = 10^5\ Hz$ & $d_2 = \dfrac{\lambda_2}{4}$, region 1 & 3 are air regions and region 2 is a lossless region with parameters $\mu_2 = \mu_0$, $\varepsilon_1 = 9\,\varepsilon_0$ & $\dfrac{\sigma}{\omega\,\varepsilon} = 0$

(i) find the values of β_1 , β_2 , β_3 , λ_2 & $\hat{\eta}_2$, (note : $\hat{\eta}_1 = \hat{\eta}_3 = 120\,\pi\ \Omega$ and $\alpha_1 = \alpha_2 = \alpha_3 = 0$) **(4 marks)**

(ii) starting with $\hat{\Gamma}_3(z) = 0$ for the rightmost region , i.e., region 3 , and using continuous $\hat{Z}$ at the interface as well as the equations in (a) , find the values of $\hat{Z}_3(0)$, $\hat{Z}_2(0)$, $\hat{\Gamma}_2(0)$, $\hat{\Gamma}_2(-10\ cm)$, $\hat{Z}_2(-10\ cm)$, $\hat{Z}_1(0)$ & $\hat{\Gamma}_1(0)$ **(10 marks)**

(iii) find the value of $\hat{E}^{-}_{m1}$ *if given* $\hat{E}^{+}_{m1} = 100\, e^{j\,50^0}\ \dfrac{V}{m}$ **(2 marks)**

Useful informations

$$e = 1.6\times10^{-19}\ C$$

$$m_e = 9.1\times10^{-31}\ kg$$

$$\mu_0 = 4\,\pi\times10^{-7}\ \frac{H}{m}$$

$$\varepsilon_0 = 8.85\times10^{-12}\ \frac{F}{m}$$

$\hat{\gamma} = \alpha + i\,\beta$ where

$$\alpha = \frac{\omega\sqrt{\mu\,\varepsilon}}{\sqrt{2}}\sqrt{\sqrt{1+\left(\frac{\sigma}{\omega\,\varepsilon}\right)^2}-1}$$

$$\beta = \frac{\omega\sqrt{\mu\,\varepsilon}}{\sqrt{2}}\sqrt{\sqrt{1+\left(\frac{\sigma}{\omega\,\varepsilon}\right)^2}+1}$$

$$\frac{1}{\sqrt{\mu_0\,\varepsilon_0}} = 3\times10^{8}\ \frac{m}{s}$$

$$\hat{\eta} = \frac{\sqrt{\frac{\mu}{\varepsilon}}}{\sqrt[4]{1+\left(\frac{\sigma}{\omega\,\varepsilon}\right)^2}}\,e^{i\frac{1}{2}\tan^{-1}\left(\frac{\sigma}{\omega\,\varepsilon}\right)}$$

$$\eta_0 = 120\,\pi\ \ \Omega = 377\ \ \Omega$$

$$\beta_0 = \omega\sqrt{\mu_0\,\varepsilon_0}$$

$$\oiint_S \vec{E}\bullet d\vec{s} = \frac{1}{\varepsilon}\iiint_V \rho_v\,dv$$

$$\oiint_S \vec{B}\bullet d\vec{s} \equiv 0$$

$$\oint_L \vec{E}\bullet d\vec{l} = -\frac{\partial}{\partial t}\left(\iint_S \vec{B}\bullet d\vec{s}\right)$$

$$\oint_L \vec{B}\bullet d\vec{l} = \mu\iint_S \vec{J}\bullet d\vec{s} + \mu\,\varepsilon\,\frac{\partial}{\partial t}\left(\iint_S \vec{E}\bullet d\vec{s}\right)$$

$$\vec{\nabla} \bullet \vec{E} = \frac{\rho_v}{\varepsilon}$$

$$\vec{\nabla} \bullet \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \times \vec{B} = \mu \vec{J} + \mu \varepsilon \frac{\partial \vec{E}}{\partial t}$$

$$\vec{J} = \sigma \vec{E}$$

$$\oiint_S \vec{F} \bullet d\vec{s} \equiv \oiiint_V \left(\vec{\nabla} \bullet \vec{F}\right) dv \qquad \textit{divergence theorem}$$

$$\oint_L \vec{F} \bullet d\vec{l} \equiv \iint_S \left(\vec{\nabla} \times \vec{F}\right) \bullet d\vec{s} \qquad \textit{Stokes' theorem}$$

$$\vec{\nabla} \bullet \left(\vec{\nabla} \times \vec{F}\right) \equiv 0$$

$$\vec{\nabla} \times \left(\vec{\nabla} f\right) \equiv 0$$

$$\vec{\nabla} \times \left(\vec{\nabla} \times \vec{F}\right) \equiv \vec{\nabla}\left(\vec{\nabla} \bullet \vec{F}\right) - \nabla^2 \vec{F}$$

$$\vec{\nabla} f = \vec{e}_x \frac{\partial f}{\partial x} + \vec{e}_y \frac{\partial f}{\partial y} + \vec{e}_z \frac{\partial f}{\partial z} = \vec{e}_\rho \frac{\partial f}{\partial \rho} + \vec{e}_\phi \frac{1}{\rho}\frac{\partial f}{\partial \phi} + \vec{e}_z \frac{\partial f}{\partial z}$$

$$= \vec{e}_r \frac{\partial f}{\partial r} + \vec{e}_\theta \frac{1}{r}\frac{\partial f}{\partial \theta} + \vec{e}_\phi \frac{1}{r \sin(\theta)}\frac{\partial f}{\partial \phi}$$

$$\vec{\nabla} \bullet \vec{F} = \frac{\partial(F_x)}{\partial x} + \frac{\partial(F_y)}{\partial y} + \frac{\partial(F_z)}{\partial z} = \frac{1}{\rho}\frac{\partial(F_\rho\, \rho)}{\partial \rho} + \frac{1}{\rho}\frac{\partial(F_\phi)}{\partial \phi} + \frac{\partial(F_z)}{\partial z}$$

$$= \frac{1}{r^2}\frac{\partial(F_r\, r^2)}{\partial r} + \frac{1}{r\sin(\theta)}\frac{\partial(F_\theta \sin(\theta))}{\partial \theta} + \frac{1}{r \sin(\theta)}\frac{\partial(F_\phi)}{\partial \phi}$$

$$\vec{\nabla} \times \vec{F} = \vec{e}_x \left(\frac{\partial(F_z)}{\partial y} - \frac{\partial(F_y)}{\partial z}\right) + \vec{e}_y \left(\frac{\partial(F_x)}{\partial z} - \frac{\partial(F_z)}{\partial x}\right) + \vec{e}_z \left(\frac{\partial(F_y)}{\partial x} - \frac{\partial(F_x)}{\partial y}\right)$$

$$= \frac{\vec{e}_\rho}{\rho}\left(\frac{\partial(F_z)}{\partial \phi} - \frac{\partial(F_\phi\, \rho)}{\partial z}\right) + \vec{e}_\phi \left(\frac{\partial(F_\rho)}{\partial z} - \frac{\partial(F_z)}{\partial \rho}\right) + \frac{\vec{e}_z}{\rho}\left(\frac{\partial(F_\phi\, \rho)}{\partial \rho} - \frac{\partial(F_\rho)}{\partial \phi}\right)$$

$$= \frac{\vec{e}_r}{r^2 \sin(\theta)}\left(\frac{\partial(F_\phi\, r \sin(\theta))}{\partial \theta} - \frac{\partial(F_\theta\, r)}{\partial \phi}\right) + \frac{\vec{e}_\theta}{r \sin(\theta)}\left(\frac{\partial(F_r)}{\partial \phi} - \frac{\partial(F_\phi\, r\sin(\theta))}{\partial r}\right) + \frac{\vec{e}_\phi}{r}\left(\frac{\partial(F_\theta\, r)}{\partial r} - \frac{\partial(F_r)}{\partial \theta}\right)$$

where $\vec{F} = \vec{e}_x F_x + \vec{e}_y F_y + \vec{e}_z F_z = \vec{e}_\rho F_\rho + \vec{e}_\phi F_\phi + \vec{e}_z F_z = \vec{e}_r F_r + \vec{e}_\theta F_\theta + \vec{e}_\phi F_\phi$ and

$d\vec{l} = \vec{e}_x\, dx + \vec{e}_y\, dy + \vec{e}_z\, dz = \vec{e}_\rho\, d\rho + \vec{e}_\phi\, \rho\, d\phi + \vec{e}_z\, dz = \vec{e}_r\, dr + \vec{e}_\theta\, r\, d\theta + \vec{e}_\phi\, r \sin(\theta)\, d\phi$

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial x^2} = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial f}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2}$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial f}{\partial r}\right) + \frac{1}{r^2 \sin(\theta)}\frac{\partial}{\partial \theta}\left(\sin(\theta)\frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2 \sin^2(\theta)}\frac{\partial^2 f}{\partial \phi^2}$$