UNIVERSITY OF SWAZILAND

102

FACULTY OF SCIENCE

DEPARTMENT OF PHYSICS

SUPPLEMENTARY EXAMINATION 2011/2012

TITLE OF PAPER	:	MATHEMATICAL METHODS FOR PHYSICISTS
COURSE NUMBER	:	P272
TIME ALLOWED	:	THREE HOURS
INSTRUCTIONS	:	ANSWER ANY FOUR OUT OF FIVE QUESTIONS. EACH QUESTION CARRIES 25 MARKS. MARKS FOR DIFFERENT SECTIONS ARE SHOWN IN THE RIGHT-HAND MARGIN.

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P272 MATHEMATICAL METHODS FOR PHYSICIST

Question one

(a) (i) Given P (-3 , - 8 , - 5) in Cartesian coordinate system, find its cylindrical and spherical coordinates. **(5 marks)**

(ii) Given P (5 , 300^0 , - 9) in cylindrical coordinate system, find its Cartesian and spherical coordinates. **(5 marks)**

(b) Given $f = \rho^2 \cos\phi - 4 z^2$,

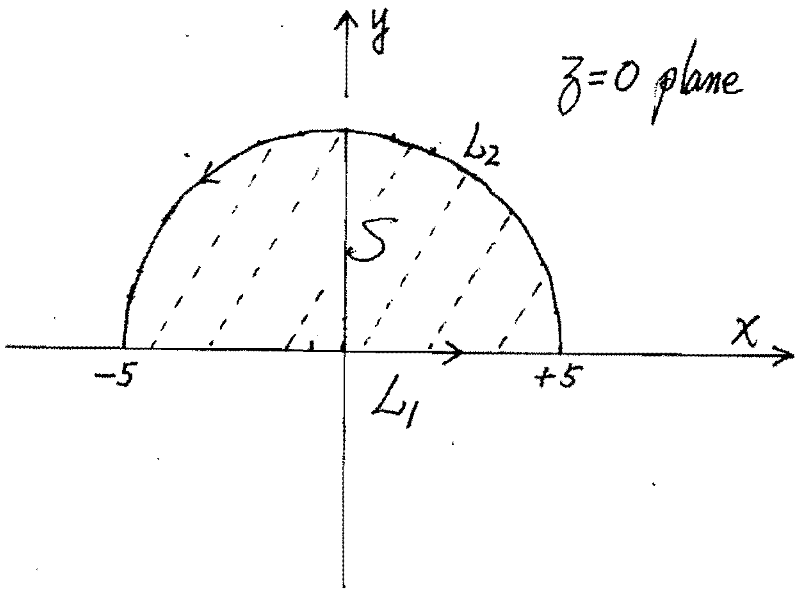
(i) find the value of $\vec{\nabla} f$ at a point P (1 , 30^0 , - 2) , **(5 marks)**

(ii) find the value of the directional derivative of f at a point P (1 , 30^0 , - 2) along the direction of $\vec{e}_\rho\, 3 - \vec{e}_\phi\, 4 + \vec{e}_z$, **(5 marks)**

(iii) find $\vec{\nabla} \times \left(\vec{\nabla} f\right)$ and shows that it is zero. **(5 marks)**

Question two

Given a vector field $\vec{F} = \vec{e}_x\left(x^2 - 2y^2\right) + \vec{e}_y\left(6xz\right) + \vec{e}_z\left(z^2 + 3y\right)$ in Cartesian system and a semi-circular closed loop $L \;\; \left(= L_1 + L_2\right)$ in counter clockwise sense on $z = 0$ plane as shown in the following diagram :



(a) Find the value of $\oint_L \vec{F} \bullet d\vec{l}$. **(10 marks)**

(Hint : $\begin{cases} L_1 & : \; z = 0 \;, \; y = 0 \; \& \; -5 \le x \le 5 \\ L_2 & : \; z = 0 \;, \; x = 5\cos(t) \;, \; y = 5\cos(t) \; \& \; 0 \le t \le \pi \end{cases}$)

(b) Find $\vec{\nabla} \times \vec{F}$ and then evaluate the value of the surface integral $\iint_S \left(\vec{\nabla} \times \vec{F}\right) \bullet d\vec{s}$ where S is bounded by the given L . Compare this answer to that obtained in (a) and make a brief comment. **(15 marks)**

(Hint : $d\vec{s} = \vec{e}_z \, dx \, dy$ and integrate about x first from $-\sqrt{25 - y^2}$ to $\sqrt{25 - y^2}$, then integrate about y from 0 to 5)

Question three

Given the following differential equation as :

$$\frac{d^2\,y(t)}{d\,t^2} + 6\,\frac{d\,y(t)}{d\,t} + 25\,y(t) = 0$$

utilize the power series method , i.e., setting $y(x) = \sum_{n=0}^{\infty} a_n\, x^{n+s}$ *and* $a_0 \neq 0$,

(a) write down the indicial equations. Find the values of s and a_1 (setting $a_0 = 1$).

(10 marks)

(b) write down the recurrence relation. For all the appropriate values of s and a_1 found in (a), set $a_0 = 1$ and use the recurrence relation to calculate the values of a_n up to the value of a_5 . Thus write down two independent solutions in their power series forms.

(15 marks)

Question four

Given the following non-homogeneous differential equation as

$$\frac{d^2\, y(t)}{d\,t^2} + 6\,\frac{d\,y(t)}{d\,t} + 25\,y(t) = 102\cos(t) + 195\sin(2t)$$

(a) find its particular solution $y_p(t)$, **(8 marks)**

(b) find the general solution to the homogeneous part of the given equation $y_h(t)$, **(6 marks)**

(c) write down the general solution to the given non-homogeneous differential equation, then find its specific solution $y_s(t)$ if the initial conditions are given as

$$y(0) = +3 \quad \& \quad \left.\frac{d\,y(t)}{d\,t}\right|_{t=0} = -1 \quad .$$ **(11 marks)**

Question five

Given the following equations for coupled oscillator system as :

$$\begin{cases} \dfrac{d^2 x_1(t)}{dt^2} = -5\,x_1(t) + 3\,x_2(t) \\ \dfrac{d^2 x_2(t)}{dt^2} = 2\,x_1(t) - 6\,x_2(t) \end{cases}$$

(a) set $x_1(t) = X_1\, e^{i\omega t}$ & $x_2(t) = X_2\, e^{i\omega t}$, deduce the following matrix equation

$A\,X = -\,\omega^2\, X$ where $X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$ & $A = \begin{pmatrix} -5 & 3 \\ 2 & -6 \end{pmatrix}$ **(4 marks)**

(b) find the eigenfrequencies ω of the given coupled system , **(5 marks)**

(c) find the eigenvectors X of the given coupled system corresponding to each eigenfrequencies found in (b). **(6 marks)**

(d) find the normal coordinates for the given coupled system, **(7 marks)**

(e) write down the general solution of the given system. **(3 marks)**

Useful informations

The transformations between rectangular and spherical coordinate systems are :

$$\begin{cases} x = r\sin(\theta)\cos(\phi) \\ y = r\sin(\theta)\sin(\phi) \\ z = r\cos(\theta) \end{cases} \quad \& \quad \begin{cases} r = \sqrt{x^2+y^2+z^2} \\ \theta = \tan^{-1}\left(\dfrac{\sqrt{x^2+y^2}}{z}\right) \\ \phi = \tan^{-1}\left(\dfrac{y}{x}\right) \end{cases}$$

The transformations between rectangular and cylindrical coordinate systems are :

$$\begin{cases} x = \rho\cos(\phi) \\ y = \rho\sin(\phi) \\ z = z \end{cases} \quad \& \quad \begin{cases} \rho = \sqrt{x^2+y^2} \\ \phi = \tan^{-1}\left(\dfrac{y}{x}\right) \\ z = z \end{cases}$$

$$\vec{\nabla} f = \vec{e}_1 \frac{1}{h_1}\frac{\partial f}{\partial u_1} + \vec{e}_2 \frac{1}{h_2}\frac{\partial f}{\partial u_2} + \vec{e}_3 \frac{1}{h_3}\frac{\partial f}{\partial u_3}$$

$$\vec{\nabla} \bullet \vec{F} = \frac{1}{h_1 h_2 h_3}\left(\frac{\partial(F_1 h_2 h_3)}{\partial u_1} + \frac{\partial(F_2 h_1 h_3)}{\partial u_2} + \frac{\partial(F_3 h_1 h_2)}{\partial u_3}\right)$$

$$\vec{\nabla} \times \vec{F} = \frac{\vec{e}_1}{h_2 h_3}\left(\frac{\partial(F_3 h_3)}{\partial u_2} - \frac{\partial(F_2 h_2)}{\partial u_3}\right) + \frac{\vec{e}_2}{h_1 h_3}\left(\frac{\partial(F_1 h_1)}{\partial u_3} - \frac{\partial(F_3 h_3)}{\partial u_1}\right) + \frac{\vec{e}_3}{h_1 h_2}\left(\frac{\partial(F_2 h_2)}{\partial u_1} - \frac{\partial(F_1 h_1)}{\partial u_2}\right)$$

where $\vec{F} = \vec{e}_1 F_1 + \vec{e}_2 F_2 + \vec{e}_3 F_3$ and

(u_1, u_2, u_3)	represents	(x, y, z)	for rectangular coordinate system
	represents	(ρ, ϕ, z)	for cylindrical coordinate system
	represents	(r, θ, ϕ)	for spherical coordinate system
$(\vec{e}_1, \vec{e}_2, \vec{e}_3)$	represents	$(\vec{e}_x, \vec{e}_y, \vec{e}_z)$	for rectangular coordinate system
	represents	$(\vec{e}_\rho, \vec{e}_\phi, \vec{e}_z)$	for cylindrical coordinate system
	represents	$(\vec{e}_r, \vec{e}_\theta, \vec{e}_\phi)$	for spherical coordinate system
(h_1, h_2, h_3)	represents	$(1, 1, 1)$	for rectangular coordinate system
	represents	$(1, \rho, 1)$	for cylindrical coordinate system
	represents	$(1, r, r\sin(\theta))$	for spherical coordinate system