140

UNIVERSITY OF SWAZILAND

FACULTY OF SCIENCE

DEPARTMENT OF PHYSICS

SUPPLEMENTARY EXAMINATION 2010/2011

TITLE OF PAPER	:	ELECTROMAGNETIC THEORY I
COURSE NUMBER	:	P331
TIME ALLOWED	:	THREE HOURS
INSTRUCTIONS	:	ANSWER ANY FOUR OUT OF FIVE QUESTIONS. EACH QUESTION CARRIES 25 MARKS. MARKS FOR DIFFERENT SECTIONS ARE SHOWN IN THE RIGHT-HAND MARGIN.

THIS PAPER HAS NINE PAGES, INCLUDING THIS PAGE.

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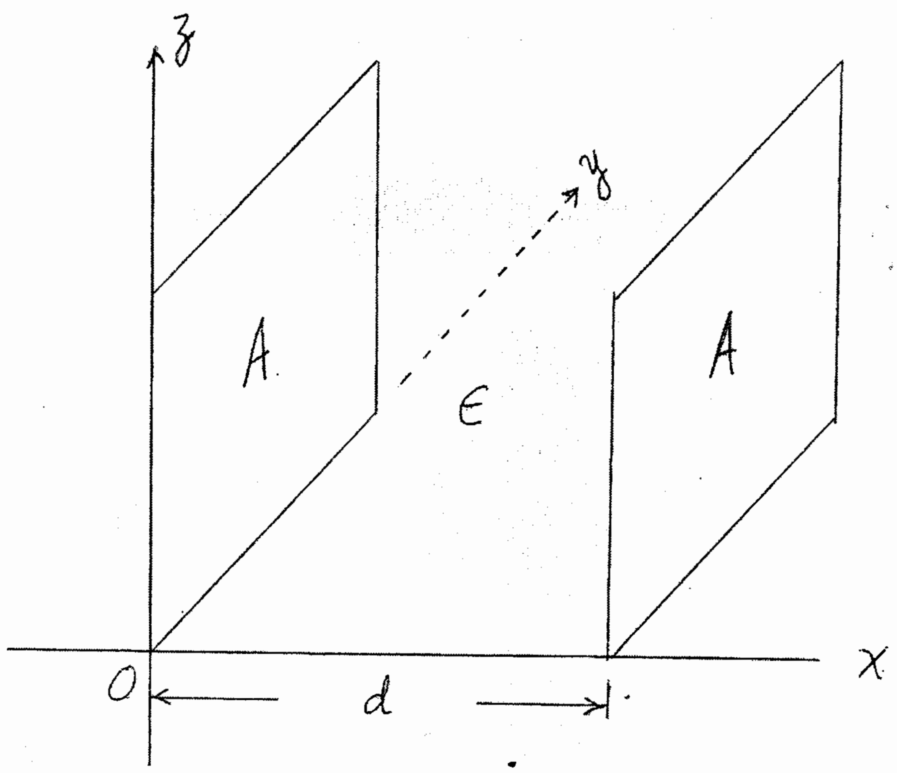
Question one

(a) For a two-parallel conducting plate capacitor system as shown in the following diagram



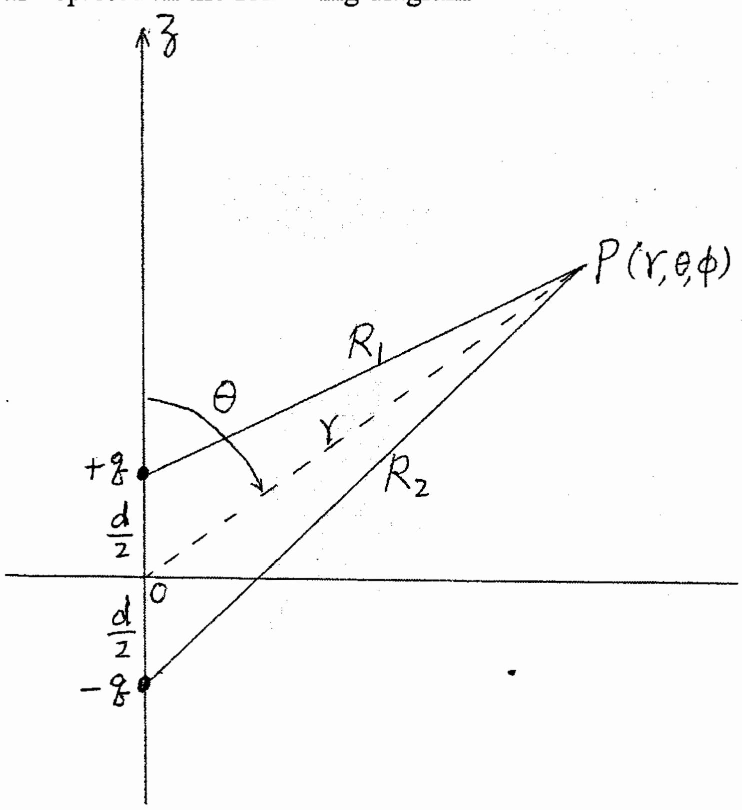
where A is the plate surface area, d is the plate separation and ε is the dielectric constant of the insulating material layer in-between the two plates.

(i) From $\nabla^2 f(x) = 0$ with boundary conditions of $f(x=0)=0$ and $f(x=d)=V_0$, find the specific solution of $f(x)$, **(4 marks)**

(ii) find $\vec{E}$ from $f(x)$ obtained in (a)(i), **(3 marks)**

(iii) find ρ_s on $x=0$ and $x=d$ respectively. Then find the total charges on both surfaces with surface area A and show that they are equal and opposite. **(5 marks)**

(b) For an electric dipole situated at the origin, i.e., $+q$ situated at $z=\frac{d}{2}$ & $-q$ situated at $z=-\frac{d}{2}$ along z – axis, the electric potential Φ at the space point $P:(r,\theta,\phi)$ due to this dipole is $\Phi=\Phi_1+\Phi_2$ where $\Phi_1=\frac{+q}{4\pi\varepsilon_0 R_1}$ and $\Phi_2=\frac{-q}{4\pi\varepsilon_0 R_2}$, as depicted in the following diagram



(i) for $r >> d$, use $R_1 \approx r-\frac{d}{2}\cos(\theta)$ and $R_2 \approx r+\frac{d}{2}\cos(\theta)$, deduce that $\Phi \approx \frac{p\cos(\theta)}{4\pi\varepsilon_9 r^2}$ where $p=q\,d$ is the electric dipole moment.

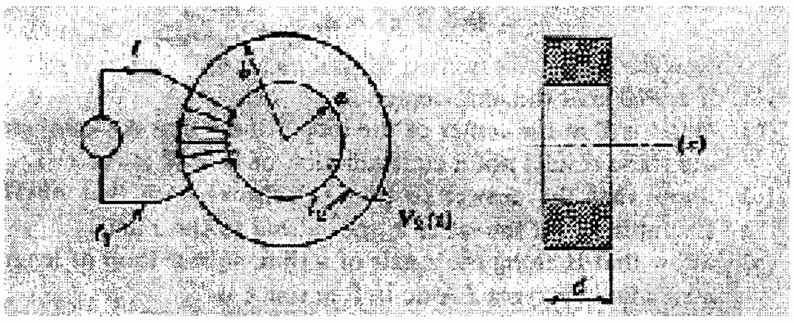
(7 marks)

(ii) from Φ in (b)(i), deduce that the electric field at far away point from the electric dipole source situated at the origin given in the above diagram is

$$\vec{E}=\frac{p}{4\pi\varepsilon_0 r^3}\left(\vec{e}_r\,2\cos(\theta)+\vec{e}_\theta\sin(\theta)\right)$$ **(6 marks)**

Questin two

(a) A static current I flows in the n_1 turn toroid l_1 wiring around a iron ring core of magnetic permeability μ with the rectangular cross-section area $(b-a)\times d$ as shown below



(i) use integral Ampere's law , choose proper closed loops to find $\vec{B}$ in terms of ρ , n_1 , μ & I within the iron core , i.e., $a \le \rho \le b$ & $0 \le z \le d$ region,

(7 marks)

(ii) find the total magnetic flux Ψ_m passing through the cross-section area $(b-a)\times d$ of the iron ring in counter clockwise sense, i.e., $\int_S \vec{B} \bullet d\vec{s}$ where $S: a \le \rho \le b$, $0 \le z \le d$ & $d\vec{s} = \vec{a}_\phi \, d\rho \, dz$, in terms of a, b, d, n_1, μ & I .

(6 marks)

(iii) find the external self-inductance L_e of the toroid wire l_1 in terms of a, b, d , μ & n_1.

(3 marks)

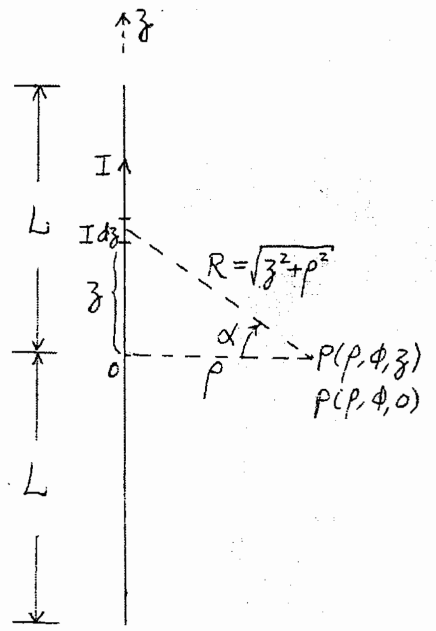
(b) Placing a single turn secondary coil l_2 around the iron ring , if the toroid wire l_1 carries a sinusoidal current of $I_0 \sin(\omega t)$ instead of carrying a static current I , find the induced e.m.f. $V_2(t)$ for a single turn secondary coil l_2 in terms of a, b, d, ω, n_1 , μ & I_0 under quasi static situation . If $a = 7\ cm$, $b = 10\ cm$, $d = 2\ cm$, $n_1 = 100$, $f = 300\ Hz$, $\mu = 4000\,\mu_0$ and $I_0 = 2\ A$, compute the amplitude of $V_2(t)$.

(9 marks)

Question three 144

(a) For the time-independent (or static) case , setting $\vec{B} = \vec{\nabla} \times \vec{A}$ and using Coulomb's gauge , i.e., $\vec{\nabla} \bullet \vec{A} = 0$, to deduce the following Poisson's equations for $\vec{A}$ from Maxwell's equations as $\nabla^2 \vec{A} = -\mu \vec{J}$. **(5 marks)**

(b) A straight thin conducting wire of length 2L carries a total current I along +z direction as shown in the diagram below :



(i) find the z-component of vector potential $\vec{A}$, i.e., A_z , due to the above given current at a point $P : (\rho, \phi, 0)$ by evaluating the following integral

$$A_z = \int_{z=-L}^{L} \frac{\mu_0 \, I \, dz}{4 \pi R} = 2 \int_{z=0}^{L} \frac{\mu_0 \, I \, dz}{4 \pi \sqrt{z^2 + \rho^2}}$$ and show that

$$A_z = \frac{\mu_0 \, I}{2 \pi} \left(\ln \left(\frac{\sqrt{\rho^2 + L^2} + L}{\rho} \right) \right)$$ **(10 marks)**

(Hint : $d \tan(\alpha) = \sec^2(\alpha) \, d\alpha$ & $1 + \tan^2(\alpha) = \sec^2(\alpha)$; $\int \sec(\alpha) \, d\alpha = \ln(\sec(\alpha) + \tan(\alpha))$)

(ii) From $\vec{B} = \vec{\nabla} \times \vec{A}$ and knowing $A_\rho = 0 = A_\phi$, find the magnetic field $\vec{B}$ at the point P due to the above given current. In the case of $L >> \rho$, show that magnetic field $\vec{B}$ obtained here can be simplified to $\vec{B} = \vec{a}_\rho \dfrac{\mu_0 \, I}{2 \pi \rho}$. **(10 marks)**

Question four

(a) Starting with the following time harmonic Maxwell's equations for a material region represented by parameters of μ , ε & σ as

$$\begin{cases} \vec{\nabla} \bullet \vec{\hat{E}}(\vec{r}) = 0 & \cdots\cdots \quad (1) \\ \vec{\nabla} \bullet \vec{\hat{H}}(\vec{r}) = 0 & \cdots\cdots \quad (2) \\ \vec{\nabla} \times \vec{\hat{E}}(\vec{r}) = -i\,\omega\,\mu\,\vec{\hat{H}}(\vec{r}) & \cdots\cdots \quad (3) \\ \vec{\nabla} \times \vec{\hat{H}}(\vec{r}) = (\sigma + i\,\omega\,\varepsilon)\,\vec{\hat{E}}(\vec{r}) & \cdots\cdots \quad (4) \end{cases}$$

and further assuming that $\vec{\hat{E}}(\vec{r})$ & $\vec{\hat{H}}(\vec{r})$ are functions of z only, i.e.,

$$\vec{\hat{E}}(\vec{r}) = \vec{a}_x\,\hat{E}_x(z) + \vec{a}_y\,\hat{E}_y(z) + \vec{a}_z\,\hat{E}_z(z) \quad \cdots\cdots \quad (5)$$

$$\vec{\hat{H}}(\vec{r}) = \vec{a}_x\,\hat{H}_x(z) + \vec{a}_y\,\hat{H}_y(z) + \vec{a}_z\,\hat{H}_z(z) \quad \cdots\cdots \quad (6)$$

deduce that $\hat{E}_z(z) = 0 = \hat{H}_z(z)$ and

$$\frac{d^2\,\hat{E}_x(z)}{d\,z^2} = -\,\hat{\gamma}^2\,\hat{E}_x(z) \quad where \quad \hat{\gamma} = \sqrt{\left(\omega^2\,\mu\,\varepsilon - i\,\omega\,\mu\,\sigma\right)} \quad ,$$

(12 marks)

(b) An uniform plane wave travelling along $+\,z$ direction with the field components of (E_x, H_y) has a complex electric field amplitude of $80\,e^{i\,40^0}\ \frac{V}{m}$ and propagates at $f = 6 \times 10^7\ Hz$ in a material region having the parameters of $\mu = \mu_0$, $\varepsilon = 4\,\varepsilon_0$ & $\frac{\sigma}{\omega\,\varepsilon} = 0.6$,

(i) Find the values of the propagation constant $\hat{\gamma}$ $(= \alpha + i\,\beta)$ and the intrinsic wave impedance $\hat{\eta}$ for this wave . **(5 marks)**

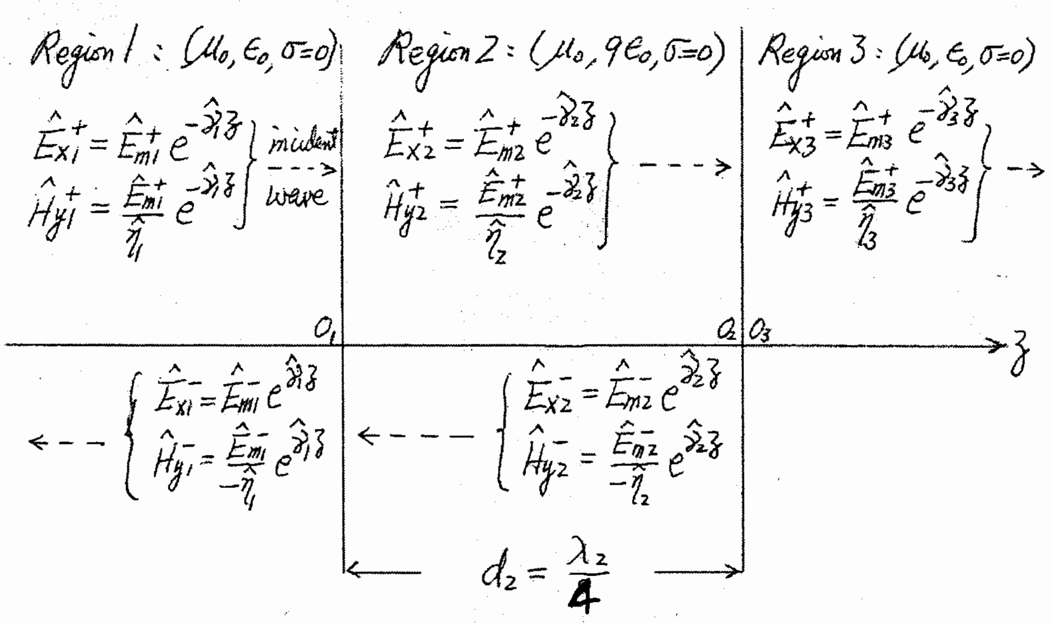
(ii) express the electric and magnetic fields in both their complex and real-time forms, with the numerical values of (b)(i) inserted , **(5 marks)**

(iii) find the values of the penetration depth , wave length and phase velocity of the given wave. **(3 marks)**

Question five

146

An uniform plane wave $\left(\hat{E}_{x1}^{+}, \hat{H}_{y1}^{+}\right)$, operates at $f = 10^{8}\ Hz$, is normally incident upon a lossless layer of $d_2 \left(= \frac{\lambda_2}{4}\right)$ thickness with parameters of $(\mu = \mu_0 , \varepsilon = 9\,\varepsilon_0)$ as shown below :



0_1 , 0_2 & 0_3 are the respective origins for region 1 , 2 & 3 chosen at the first and second interface . (Both region 1 and region 3 are air region.)

(a) Define for the i^{th} *region* $(i = 1,2,3)$ the reflection coefficient $\hat{\Gamma}_i(z)$ and the total wave impedance $\hat{Z}_i(z)$ and deduce the following :

$$\begin{cases} \hat{Z}_i(z) = \hat{\eta}_i \dfrac{1+\hat{\Gamma}_i(z)}{1-\hat{\Gamma}_i(z)} \quad \text{and reversely} \quad \hat{\Gamma}_i(z) = \dfrac{\hat{Z}_i(z)-\hat{\eta}_i}{\hat{Z}_i(z)+\hat{\eta}_i} \quad \text{and} \\ \hat{\Gamma}_i(z') = \hat{\Gamma}_i(z)\, e^{2\hat{\gamma}_i (z'-z)} \quad \text{where } z' \ \& \ z \text{ are two points in } i^{th} \text{ region} \end{cases}$$

(10 marks)

(b) (i) find the values of $\hat{\gamma}_1$, $\hat{\gamma}_2$, $\hat{\gamma}_3$, $\hat{\eta}_2$ & λ_2 , (note : $\hat{\eta}_1 = \hat{\eta}_3 = 120\,\pi\ \Omega$ and $\alpha_1 = \alpha_2 = \alpha_3 = 0$) **(3 marks)**

(ii) starting with $\hat{\Gamma}_3(z) = 0$ for the rightmost region , i.e., region 3 , and using continuous $\hat{Z}$ at the interface as well as the equations in (a) , find the values of $\hat{Z}_3(0)$, $\hat{Z}_2(0)$, $\hat{\Gamma}_2(0)$, $\hat{\Gamma}_2(-d_2)$, $\hat{Z}_2(-d_2)$, $\hat{Z}_1(0)$ & $\hat{\Gamma}_1(0)$

(10 marks)

(iii) find the value of $\hat{E}_{m1}^{-}$ *if given* $\hat{E}_{m1}^{+} = 50\, e^{j\,40^0}\ \frac{V}{m}$ **(2 marks)**

Useful informations

$$e = 1.6 \times 10^{-19}\ C$$

$$m_e = 9.1 \times 10^{-31}\ kg$$

$$\mu_0 = 4\,\pi \times 10^{-7}\ \frac{H}{m}$$

$$\varepsilon_0 = 8.85 \times 10^{-12}\ \frac{F}{m}$$

$$\alpha = \frac{\omega\sqrt{\mu\,\varepsilon}}{\sqrt{2}}\sqrt{\sqrt{1+\left(\frac{\sigma}{\omega\,\varepsilon}\right)^2}-1}$$

$$\beta = \frac{\omega\sqrt{\mu\,\varepsilon}}{\sqrt{2}}\sqrt{\sqrt{1+\left(\frac{\sigma}{\omega\,\varepsilon}\right)^2}+1}$$

$$\frac{1}{\sqrt{\mu_0\,\varepsilon_0}} = 3 \times 10^8\ \frac{m}{s}$$

$$\hat{\eta} = \frac{\sqrt{\frac{\mu}{\varepsilon}}}{\sqrt[4]{1+\left(\frac{\sigma}{\omega\,\varepsilon}\right)^2}}\, e^{\,i\frac{1}{2}\tan^{-1}\left(\frac{\sigma}{\omega\,\varepsilon}\right)}$$

$$\eta_0 = 120\,\pi\ \ \Omega = 377\ \ \Omega$$

$$\beta_0 = \omega\sqrt{\mu_0\,\varepsilon_0}$$

$$\oiint_S \vec{E} \bullet d\vec{s} = \frac{1}{\varepsilon}\iiint_V \rho_v\, dv$$

$$\oiint_S \vec{B} \bullet d\vec{s} \equiv 0$$

$$\oint_L \vec{E} \bullet d\vec{l} = -\frac{\partial}{\partial t}\left(\iint_S \vec{B} \bullet d\vec{s}\right)$$

$$\oint_L \vec{B} \bullet d\vec{l} = \mu \iint_S \vec{J} \bullet d\vec{s} + \mu\varepsilon\frac{\partial}{\partial t}\left(\iint_S \vec{E} \bullet d\vec{s}\right)$$

$$\vec{\nabla} \bullet \vec{E} = \frac{\rho_v}{\varepsilon}$$

$$\vec{\nabla} \bullet \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \times \vec{B} = \mu\,\vec{J} + \mu\,\varepsilon\frac{\partial \vec{E}}{\partial t}$$

$$\vec{J} = \sigma\,\vec{E}$$

$$\oint\!\!\oint_S \vec{F} \bullet d\,\vec{s} \equiv \oint\!\!\oint\!\!\oint_V \left(\vec{\nabla} \bullet \vec{F}\right) d\,v \qquad \textit{divergence theorem}$$

$$\oint_L \vec{F} \bullet d\,\vec{l} \equiv \iint_S \left(\vec{\nabla} \times \vec{F}\right) \bullet d\,\vec{s} \qquad \textit{Stokes' theorem}$$

$$\vec{\nabla} \bullet \left(\vec{\nabla} \times \vec{F}\right) \equiv 0$$

$$\vec{\nabla} \times \left(\vec{\nabla}\, f\right) \equiv 0$$

$$\vec{\nabla} \times \left(\vec{\nabla} \times \vec{F}\right) \equiv \vec{\nabla}\left(\vec{\nabla} \bullet \vec{F}\right) - \nabla^2\, \vec{F}$$

$$\vec{\nabla}\, f = \vec{e}_x \frac{\partial f}{\partial x} + \vec{e}_y \frac{\partial f}{\partial y} + \vec{e}_z \frac{\partial f}{\partial z} = \vec{e}_\rho \frac{\partial f}{\partial \rho} + \vec{e}_\phi \frac{1}{\rho}\frac{\partial f}{\partial \phi} + \vec{e}_z \frac{\partial f}{\partial z}$$

$$= \vec{e}_r \frac{\partial f}{\partial r} + \vec{e}_\theta \frac{1}{r}\frac{\partial f}{\partial \theta} + \vec{e}_\phi \frac{1}{r\sin(\theta)}\frac{\partial f}{\partial \phi}$$

$$\vec{\nabla} \bullet \vec{F} = \frac{\partial(F_x)}{\partial x} + \frac{\partial(F_y)}{\partial y} + \frac{\partial(F_z)}{\partial z} = \frac{1}{\rho}\frac{\partial(F_\rho\, \rho)}{\partial \rho} + \frac{1}{\rho}\frac{\partial(F_\phi)}{\partial \phi} + \frac{\partial(F_z)}{\partial z}$$

$$= \frac{1}{r^2}\frac{\partial(F_r\, r^2)}{\partial r} + \frac{1}{r\sin(\theta)}\frac{\partial(F_\theta \sin(\theta))}{\partial \theta} + \frac{1}{r\sin(\theta)}\frac{\partial(F_\phi)}{\partial \phi}$$

$$\vec{\nabla} \times \vec{F} = \vec{e}_x \left(\frac{\partial(F_z)}{\partial y} - \frac{\partial(F_y)}{\partial z}\right) + \vec{e}_y \left(\frac{\partial(F_x)}{\partial z} - \frac{\partial(F_z)}{\partial x}\right) + \vec{e}_z \left(\frac{\partial(F_y)}{\partial x} - \frac{\partial(F_x)}{\partial y}\right)$$

$$= \frac{\vec{e}_\rho}{\rho}\left(\frac{\partial(F_z)}{\partial \phi} - \frac{\partial(F_\phi\, \rho)}{\partial z}\right) + \vec{e}_\phi \left(\frac{\partial(F_\rho)}{\partial z} - \frac{\partial(F_z)}{\partial \rho}\right) + \frac{\vec{e}_z}{\rho}\left(\frac{\partial(F_\phi\, \rho)}{\partial \rho} - \frac{\partial(F_\rho)}{\partial \phi}\right)$$

$$= \frac{\vec{e}_r}{r^2\sin(\theta)}\left(\frac{\partial(F_\phi\, r\sin(\theta))}{\partial \theta} - \frac{\partial(F_\theta\, r)}{\partial \phi}\right) + \frac{\vec{e}_\theta}{r\sin(\theta)}\left(\frac{\partial(F_r)}{\partial \phi} - \frac{\partial(F_\phi\, r\sin(\theta))}{\partial r}\right) + \frac{\vec{e}_\phi}{r}\left(\frac{\partial(F_\theta\, r)}{\partial r} - \frac{\partial(F_r)}{\partial \theta}\right)$$

where $\vec{F} = \vec{e}_x\, F_x + \vec{e}_y\, F_y + \vec{e}_z\, F_z = \vec{e}_\rho\, F_\rho + \vec{e}_\phi\, F_\phi + \vec{e}_z\, F_z = \vec{e}_r\, F_r + \vec{e}_\theta\, F_\theta + \vec{e}_\phi\, F_\phi$ and

$$d\,\vec{l} = \vec{e}_x\, dx + \vec{e}_y\, dy + \vec{e}_z\, dz = \vec{e}_\rho\, d\rho + \vec{e}_\phi\, \rho\, d\phi + \vec{e}_z\, dz = \vec{e}_r\, dr + \vec{e}_\theta\, r\, d\theta + \vec{e}_\phi\, r\sin(\theta)\, d\phi$$

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial x^2} = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial f}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2}$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin(\theta)}\frac{\partial}{\partial \theta}\left(\sin(\theta)\frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2\sin^2(\theta)}\frac{\partial^2 f}{\partial \phi^2}$$