UNIVERSITY OF SWAZILAND

FACULTY OF SCIENCE

DEPARTMENT OF PHYSICS

SUPPLEMENTARY EXAMINATION 2007/2008

TITLE OF PAPER	**:**	**MATHEMATICAL METHODS FOR PHYSICISTS**
COURSE NUMBER	**:**	**P272**
TIME ALLOWED	**:**	**THREE HOURS**
INSTRUCTIONS	**:**	**ANSWER ANY FOUR OUT OF FIVE QUESTIONS.** **EACH QUESTION CARRIES 25 MARKS.** **MARKS FOR DIFFERENT SECTIONS ARE SHOWN IN THE RIGHT-HAND MARGIN.**

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P272 MATHEMATICAL METHODS FOR PHYSICIST

Question one

(a) (i) Given the rectangular coordinates of a point P as $(-5\ ,\ -2\ ,\ 9)$, find its cylindrical and spherical coordinates respectively . Express the answers of angles in degrees . **(4 marks)**

(ii) Given the spherical coordinates of a point P as $(3\,,75^0\,,160^0)$, find its cylindrical and rectangular coordinates respectively . **(4 marks)**

(b) For a point P on $\rho - z\ plane$, i.e., $\phi = constant$,

(i) draw the cylindrical unit vectors $\vec{e}_\rho$, $\vec{e}_z$ as well as the spherical unit vectors $\vec{e}_r$, $\vec{e}_\theta$ for the given point on $x - y\ plane$, **(3 marks)**

(ii) express $\vec{e}_r$, $\vec{e}_\theta$ in terms of $\vec{e}_\rho$, $\vec{e}_z$ and deduce that

$$\frac{\partial \vec{e}_r}{\partial \theta} = +\vec{e}_\theta \quad \text{and} \quad \frac{\partial \vec{e}_\theta}{\partial \theta} = -\vec{e}_r$$ **(5 marks)**

(c) Given $f(\rho,\phi,z) = \rho + 2\rho\sin(\phi) - 3z$,

(i) find $\vec{\nabla} f$, **(3 marks)**

(ii) evaluate $\vec{\nabla} f$ at the point $P:(7\,,210^0\,,-15)$ and also find the directional derivative of f along the direction of $\vec{e}_\rho 3 + \vec{e}_z 4$. **(6 marks)**

Question two

(a) Given any scalar function f in Cartesian coordinate system , i.e., $f = f(x,y,z)$,

verify the following identity :

$\vec{\nabla} \times \left(\vec{\nabla} f\right) \equiv 0$ **(5 marks)**

(b) Given a vector field $\vec{G}(\rho,\phi,z) = \vec{e}_\rho\, \rho^2 + \vec{e}_\phi\, 4\, e^\rho + \vec{e}_z \left(z^2 - \rho^2 \cos(\phi)\right)$,

(i) carry out the following closed line integration of $\oint_L \vec{G} \bullet d\vec{l}$

where L : a circular closed loop of radius $\mathbf{5}$ in counter clockwise sense on

$z = \mathbf{0}$ plane centred at the origin , i.e., $d\vec{l} = \vec{e}_\phi\, \rho\, d\phi \xrightarrow{\rho=5} \vec{e}_\phi\, 5\, d\phi$,

(8 marks)

(ii) find $\vec{\nabla} \times \vec{G}$. Then carry out the surface integral of $\iint_S (\vec{\nabla} \times \vec{G}) \bullet d\vec{s}$ where

S : the surface enclosed by the given closed loop in (b) (i) , i.e.,

$d\vec{s} = \vec{e}_z\, \rho\, d\rho\, d\phi$ with $\mathbf{0} \le \rho \le \mathbf{5}$ and $\mathbf{0} \le \phi \le \mathbf{2}\pi$. Compare it with

that obtained in (b)(i) and make brief comments on Stokes' theorem.

(12 marks)

Question three

If the transverse wave amplitude function $u(x,t)$ of a certain vibrating string follows the following partial differential equation : $\frac{\partial^2 u(x,t)}{\partial x^2} - \frac{1}{16}\frac{\partial^2 u(x,t)}{\partial t^2} = 0$,

(a) set $u(x,t) = X(x)\,T(t)$ and utilize the separation variable scheme to deduce the following two ordinary differential equations :

$$\begin{cases} \frac{d^2 X(x)}{dx^2} = -k^2 X(x) \\ \frac{d^2 T(t)}{dt^2} = -16\,k^2\,T(t) \end{cases}$$ where k is a separation constant , **(4 marks)**

(b) (i) by direct substitution, show that $X(x) = A_k \cos(k\,x) + B_k \sin(k\,x)$ and $T(t) = C_k\,cos(4\,k\,t) + D_k\,sin(4\,k\,t)$ are a general solution to the ordinary differential equations in (a) with A_k , B_k , C_k *and* D_k as arbitrary constants, **(3 marks)**

(ii) given the length of the vibrating string as 7 metres with both ends fixed , i.e., $u(0,t) = 0 = u(7,t)$, find the eigenvalues of k and write down the general solution of $u(x,t)$ to include all the eigenvalues of k , **(6 marks)**

Question three (continued)

(c) given the initial condition as $\left.\dfrac{\partial u(x,t)}{\partial t}\right|_{t=0} = 0$ and

$$u(x,0) = \begin{cases} \dfrac{5}{2}x & \text{for } 0 \le x \le 2 \\ -x+7 & \text{for } 2 \le x \le 7 \end{cases}$$

, determine the specific values of those arbitrary constants in the general solution of $u(x,t)$ written down in (b)(ii) and thus write down the specific solution of this given problem.

(hint : $\displaystyle\int_0^7 sin(\frac{n\pi}{7}x)\ sin(\frac{m\pi}{7}x)\ dx = \begin{cases} \dfrac{7}{2} & if \quad n = m \\ 0 & if \quad n \neq m \end{cases}$

where n and m are non-zero positive integers) **(12 marks)**

Question four

Given the following differential equation $\frac{d^2 y(x)}{dx^2} - 7 \frac{d y(x)}{dx} + 3 y(x) = 0$,

using the power series method , i.e., set $y(x) = \sum_{n=0}^{\infty} a_n x^{n+s}$ with $a_0 \neq 0$ and substituting it back to the given differential equation ,

(a) requiring the coefficients of the lowest power terms for x , i.e., x^{s-2} and x^{s-1} , to be zero and thus write down the indicial equations. From these equations find the values of s (possibly also the values of a_1 from setting $a_0 = 1$), **(6 marks)**

(b) requiring the coefficients of all the rest power terms for x , i.e., x^{s+n} with $n = 0, 1, 2, 3, \cdots$, to be zero and find the recurrence relation, **(5 marks)**

(c) (i) using the recurrence relation in (b) , find the values of a_2 , a_3 ,$\cdots$ a_6 if $a_0 = 1$ for each value of s found in (a) . **(12 marks)**

(ii) write down the general solution of the given differential equation. **(2 marks)**

Question five

(a) Given $m\dfrac{d^2 x}{dt^2} = -kx$, and $m = \dfrac{1}{5}\ kg$ & $k = 8\ \dfrac{N}{m}$

(i) find the values of the angular frequency , frequency and period of the given simple harmonic oscillator system , **(3 marks)**

(ii) write down the general solution of the given problem . **(2 marks)**

(b) Two simple harmonic oscillators (one is represented by m_1 and k_1 and the other represented by m_2 and k_2) are jointed together by a spring of spring constant k_3 . The coupled differential equations are simplified to be :

$$\begin{cases} \dfrac{d^2 x_1}{dt^2} = -9x_1 + 4x_2 \\ \dfrac{d^2 x_2}{dt^2} = 3x_1 - 10x_2 \end{cases}$$

(i) set $x_1(t) = X_1 e^{i\omega t}$ and $x_2(t) = X_2 e^{i\omega t}$, deduce the following matrix equation $\lambda X = AX$

where $\lambda = -\omega^2$, $X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$ and $A = \begin{pmatrix} -9 & 4 \\ 3 & -10 \end{pmatrix}$

(4 marks)

(ii) find the eigenfrequencies ω for the matrix equation in (b)(i) **(6 marks)**

(iii) find the eigenvectors corresponding to the eigenfrequencies found in (b)(ii) respectively , **(5 marks)**

(iv) find the normal coordinates of the system . **(5 marks)**

Useful informations

The transformations between rectangular and spherical coordinate systems are :

$$\begin{cases} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin s\phi \\ z = r\cos\theta \end{cases} \qquad \begin{cases} r = \sqrt{x^2+y^2+z^2} \\ \theta = \tan^{-1}\dfrac{\sqrt{x^2+y^2}}{z} \\ \phi = \tan^{-1}\dfrac{y}{x} \end{cases}$$

The transformations between rectangular and cylindrical coordinate systems are :

$$\begin{cases} x = \rho\cos\phi \\ y = \rho\sin\phi \\ z = z \end{cases} \qquad \begin{cases} \rho = \sqrt{x^2+y^2} \\ \phi = \tan^{-1}\dfrac{y}{x} \\ z = z \end{cases}$$

$$\vec{\nabla} f = \vec{e}_1 \frac{1}{h_1}\frac{\partial f}{\partial u_1} + \vec{e}_2 \frac{1}{h_2}\frac{\partial f}{\partial u_2} + \vec{e}_3 \frac{1}{h_3}\frac{\partial f}{\partial u_3}$$

$$\vec{\nabla} \bullet \vec{F} = \frac{1}{h_1 h_2 h_3}\left(\frac{\partial(F_1 h_2 h_3)}{\partial u_1} + \frac{\partial(F_2 h_1 h_3)}{\partial u_2} + \frac{\partial(F_3 h_1 h_2)}{\partial u_3}\right)$$

$$\vec{\nabla} \times \vec{F} = \frac{\vec{e}_1}{h_2 h_3}\left(\frac{\partial(F_3 h_3)}{\partial u_2} - \frac{\partial(F_2 h_2)}{\partial u_3}\right) + \frac{\vec{e}_2}{h_1 h_3}\left(\frac{\partial(F_1 h_1)}{\partial u_3} - \frac{\partial(F_3 h_3)}{\partial u_1}\right) + \frac{\vec{e}_3}{h_1 h_2}\left(\frac{\partial(F_2 h_2)}{\partial u_1} - \frac{\partial(F_1 h_1)}{\partial u_2}\right)$$

where $\vec{F} = \vec{e}_1 F_1 + \vec{e}_2 F_2 + \vec{e}_3 F_3$ and

(u_1, u_2, u_3) represents (x, y, z) for rectangular coordinate system

represents (ρ, ϕ, z) for cylindrical coordinate system

represents (r, θ, ϕ) for spherical coordinate system

$(\vec{e}_1, \vec{e}_2, \vec{e}_3)$ represents $(\vec{e}_x, \vec{e}_y, \vec{e}_z)$ for rectangular coordinate system

represents $(\vec{e}_\rho, \vec{e}_\phi, \vec{e}_z)$ for cylindrical coordinate system

represents $(\vec{e}_r, \vec{e}_\theta, \vec{e}_\phi)$ for spherical coordinate system

(h_1, h_2, h_3) represents $(1, 1, 1)$ for rectangular coordinate system

represents $(1, \rho, 1)$ for cylindrical coordinate system

represents $(1, r, r\sin\theta)$ for spherical coordinate system