University of Eswatini

Final Examination – 2020/2021

MSc in Mathematics

Title of Paper : Asymptotic Analysis
Course Number : MAT605
Time Allowed : Three (3) hours

Instructions:

1. This paper consists of 2 sections.
2. Answer ALL questions in Section A.
3. Answer ANY 3 (out of 5) questions in Section B.
4. Show all your working.
5. Begin each question on a new page.

Section A

Answer ALL Questions in this section

A.1 a. Consider the quadratic equation

$$x^2 - (2 - \varepsilon)x - 8 = 0,$$

Find a 3-term perturbation solution of the form

$$x(\varepsilon) = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots,$$

valid when $\varepsilon << 1$. [15 marks]

b. Consider the initial value problem

$$\dot{y} - 4\varepsilon t\sqrt{y} = 0, \quad y(0) = e^2.$$

i. Find the *exact* solution of the problem. [5 marks]

ii. By letting

$$y(t;\varepsilon) = y_0(t) + \varepsilon y_1(t) + \varepsilon^2 y_2(t) + \cdots, \quad \varepsilon << 1,$$

and substituting into the problem, obtain an expression for y_0, y_1 and $y_n, n \geqslant 2$. [10 marks]

c. Obtain the leading order asymptotic approximation of the integral

$$\int_0^\infty \frac{\sin\left[\lambda(3t - t^3)\right]}{t+1}\,dt$$

valid as $\lambda \to \infty$. [10 marks]

Section B

Answer ANY 3 Questions in this section

B.2 a. Determine whether each order relation is true or false.

i. $\dfrac{x^{3/2}}{1-\cos(x^2)} = O\left(x^{-3/2}\right)$, as $x \to 0$ [5 marks]

ii. $e^{-x} = o\left(\dfrac{1}{x^N}\right)$, $\forall N > 0$ as $x \to \infty$ [5 marks]

b. Consider the nonlinear boundary value problem (BVP)

$$\ddot{y} - \varepsilon y\dot{y} - 2 = 0, \; y(0) = \dot{y}(0) = 0.$$

Find a 3-term perturbation solution of the form

$$y(t;\varepsilon) = y_0(t) + \varepsilon y_1(t) + \varepsilon^2 y_2(t) + \cdots, \quad \varepsilon << 1,$$

for the BVP. [10 marks]

B.3 Consider the cubic equation

$$\varepsilon x^3 + x - 1 = 0.$$

Find a 3-term perturbation solution of the form

$$x(\varepsilon) = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots,$$

valid when $\varepsilon << 1$. [20 marks]

B.4 Consider the boundary value problem (BVP)

$$\varepsilon y'' + y' - y = 0, \;\; y(0) = 1, \; y(1) = 1,$$

where the parameter $\varepsilon << 1$. By assuming that a *boundary layer* exists at the $x = 0$ end, find

a. the leading order term of the *outer solution* [4 marks]

b. the *distinguished limit* and hence the rescaled inner variable [6 marks]

c. the leading order term of the *inner solution* [7 marks]

d. the leading order term of the *composite solution* [3 marks]

B.5 a. *Watson's Lemma* states that, provided $\alpha > -1$, and $f(t)$ is exponentially bounded and analytic at the origin, possesing a Mclaurin series, then

$$\int_0^a e^{-\lambda t} t^\alpha f(t) \mathrm{d}t \sim \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} \frac{\Gamma(n+\alpha+1)}{\lambda^{n+\alpha+1}}, \quad \text{as } \lambda \to \infty, \tag{1}$$

where $\Gamma(n)$ is the Gamma function. Prove Watson's Lemma (**??**). [10 marks]

b. Use Watson's Lemma to find a 2-term asymptotic approximation of

$$\int_0^\infty e^{-\lambda t^2} t \sin t \, \mathrm{d}t,$$

valid as $\lambda \to \infty$. [10 marks]

B.6 a. Find the leading order asymptotic approximation of each integral as $\lambda \to \infty$.

i. $\displaystyle\int_0^2 \sqrt{1+t^2} \exp\left[\lambda t e^{1-t}\right] \mathrm{d}t$ [7 marks]

ii. $\displaystyle\int_0^\pi \frac{\cos\left(\lambda \sin t\right)}{\sqrt{1+4t^2}} \, \mathrm{d}t$ [7 marks]

b. Use the definition

$$J_n(x) = \frac{1}{\pi} \int_0^\pi \cos\left[x \sin\theta - n\theta\right] \mathrm{d}\theta$$

to show that, as $x \to \infty$, the leading order approximation of the Bessel function

$$J_n(x) \sim \sqrt{\frac{2}{\pi x}} \cos(x - \tfrac{1}{2}n\pi - \tfrac{1}{4}\pi).$$ [6 marks]

END OF EXAMINATION