University of Swaziland

Main Examination, 2019/2020

MSc.I

Title of Paper : Advanced Statistics

Course Number : MAT636

Time Allowed : Three (3) Hours

Instructions

1. This paper consists of SIX (6) questions.
2. Show all your working.
3. Start each new major question on a new page and clearly indicate the question number at the top of the page.
4. You can answer questions in any order.
5. Indicate your program next to your student ID.

Special Requirements: NONE

ANSWER ANY *FOUR* QUESTIONS

QUESTION A1 [*25 Marks*]

A1 (a) The joint probability distribution function of two random variables Y and Z is

$$P(Y=y, Z=z) = \frac{e^{-\lambda}\lambda^{y+z}}{y!z!}\theta^{y}(1-\theta)^{z}, \quad y=0,1,2,\ldots; z=0,1,2,\ldots$$

(i) Find the marginal distribution of Y and identify it.

[5 Marks]

(ii) Show that Y and Z are independent.

[8 Marks]

(b) Let $Y_1, Y_2, \ldots, Y_n$ denote independent random variables with cumulative distribution function $F(y)$ and probability density function $f(y)$.

(i) Derive the probability density function of $Y_{(n)} = max\{Y_1, Y_2, \ldots, Y_n\}$.

[7 Marks]

(ii) Electronic components of a certain type have a length of life Y, with probability density function given by

$$f(y) = \begin{cases} (1/100)e^{-y/100}, & \text{if } y \geq 0 \\ 0, & \text{elsewhere} \end{cases}$$

(Length of life is measured in hours.) Suppose that two of such components operate independently and in parallel in a certain system (hence, the system does not fail until both components fail). Find the density function for X, the length of life of the system. Hence compute the probability that $X > 200$ hours.

[5 Marks]

QUESTION A2 [*25 Marks*]

A2 (a) Let $X_1, \ldots, X_n$ be independent and identically distributed random variables with variance $\sigma^2 < \infty$. If

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X})^2$$

is the variance of a random sample from an infinite population, show that S^2 is an unbiased estimator for σ^2.

[8 Marks]

(b) The reaction of an individual to a stimulus in a psychological experiment may take one of two forms, A or B. If an experimenter wishes to estimate the probability p that a person will react in manner A, how many people must be included in the experiment? Assume that the experimenter will be satisfied if the error of estimation is less than 0.04 with probability equal to 0.90. Assume also that he expects p to lie somewhere in the neighborhood of 0.6.

[6 Marks]

(a) Let $Y_1, Y_2, \ldots, Y_n$ be independent random variables with $E(Y_i) = \mu$ and $V(Yi) = \sigma^2$. Let

$$\bar{Y} = \frac{1}{n}\sum_{i=1}^{n} Y_i$$

and show that $E(\bar{Y}) = \mu$ and $Var(\bar{Y}) = \sigma^2/n$.

[8 Marks]

QUESTION A3 [*25 Marks*]

A3 (a) Compute the median of the uniform distribution on the interval (θ_1, θ_2)?

[5 Marks]

(b) Consider the experiment of tossing a fair coin 3 times. Let X be the number of heads on the first toss and F the number of heads on the first two tosses. Fill the joint probability table for X and F. Compute Cov(X, F).

[10 Marks]

(c) A psychological study was conducted to compare the reaction times of men and women to a stimulus. Independent random samples of 50 men and 50 women were employed in the experiment.The results are shown in Table below.

Men	Women
$n_1 = 50$	$n_2 = 50$
$\bar{y}_1 = 3.6$ seconds	$\bar{y}_2 = 3.8$ seconds
$s_1^2 = 0.18$	$s_2 = 0.14$

(i) Do the data present sufficient evidence to suggest a difference between true mean reaction times for men and women? Use $\alpha = 0.05$.

[8 Marks]

(ii) Find the p-value for the statistical test.

[2 Marks]

QUESTION A4 [*25 Marks*]

A4 (a) Let $Y_1, \ldots, Y_n$ be a random sample from a population with pdf

$$fy) = \begin{cases} \frac{1}{\alpha} y^{(1-\alpha)/\alpha}, & \text{for } 0 < y < 1; \alpha > 0 \\ 0, & \text{otherwise} \end{cases}$$

(i) Show that the maximum likelihood estimator of α is $\hat{\alpha} = -(1/n)\sum_{i=1}^{n} \ln(Y_i)$.

[7 Marks]

(ii) Is $\hat{\alpha}$ a consistent estimator of α?

[8 Marks]

(b) A random sample of size 36 from a population with known variance, $\sigma^2 = 9$, yields a sample mean of $\bar{x} = 17$. Compute the type II error β for testing the hypothesis $H_0 : \mu = 15$ versus $H_a : \mu = 16$. Assume $\alpha = 0.05$.

[10 Marks]

QUESTION A5 [*25 Marks*]

A5 (a) The discrete random variable X has the binomial distribution

$$P(X = x) = \binom{m}{x} \theta^x (1-\theta)^{m-x}, \quad x = 0, 1, \ldots, m$$

where m is a positive integer and $0 < \theta < 1$. Find the moment-generating function for X and use it to find the expected value and variance.

[8 Marks]

(b) A diagnostic test for a disease is such that it (correctly) detects the disease in 90% of the individuals who actually have the disease. Also, if a person does not have the disease, the test will report that they do not have it with probability 0.9. Only 1% of the population has the disease in question. If a person is chosen at random from the population and the diagnostic test indicates that they have the disease, what is the conditional probability that the person does, in fact, have the disease?

[6 Marks]

(c) The continuous random variables X and Y have joint probability density function

$$f_{XY}(x,y) = \begin{cases} \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)} x^{\alpha-1} y^{\beta-1} (1-x-y)^{\gamma-1}, & 0 < x < 1, 0 < y < 1, x+y < 1, \\ 0, & \text{otherwise} \end{cases}$$

where $\alpha > 0$, $\beta > 0$ and $\gamma > 0$ are parameters and $\Gamma(.)$ is the gamma function. Obtain the joint probability density function of

$$U = 1 - X, \; V = \frac{Y}{1-X}.$$

[11 Marks]

QUESTION A6 [*25 Marks*]

A6 (a)

i. Let $X_1, \ldots, X_n$ be a random sample of size n from the exponential distribution whose pdf is

$$f(x, \theta) = \begin{cases} \theta e^{-\theta x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

(i) Use the method of moments to find a point estimator for θ.

[5 Marks]

(ii) The following data represent the time intervals between the emissions of beta particles: Assuming

0.9	0.1	0.1	0.8	0.9	0.1	0.1	0.7	1.0	0.2
0.1	0.1	0.1	2.3	0.8	0.3	0.2	0.1	1.0	0.9
0.1	0.5	0.4	0.6	0.2	0.4	0.2	0.1	0.8	0.2
0.5	3.0	1.0	0.5	0.2	2.0	1.7	0.1	0.3	0.1
0.4	0.5	0.8	0.1	0.1	1.7	0.1	0.2	0.3	0.1

the data follow an exponential distribution, use a(i) to compute a moment estimate for the parameter θ. Interpret.

[4 Marks]

(b) In Bayesian inference define a conjugate prior distribution.

[3 Marks]

(c) Let $Y_1, Y_2, \ldots, Y_n$ denote a random sample from a Bernoulli distribution where

$$P(Y_i = 1) = p \text{ and } P(Y_i = 0) = 1 - p,$$

and assume that the prior distribution for p is $beta(\alpha, \beta)$, i.e.

$$f(y) = \begin{cases} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} y^{\alpha-1} (1-y)^{\beta-1}, & 0 < y < 1, \\ 0, & \text{otherwise.} \end{cases}$$

(i) Find the posterior distribution for p.

[8 Marks]

(ii) Find the Bayes estimators for p.

[5 Marks]

END OF EXAMINATION PAPER

Table 1: Standard Normal Probabilities

z	0	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990