University of Eswatini

Final Examination, August 2020

MSc Applied Mathematics

Title of Paper : Special Topics in Mathematical Modelling

Course Code : MAT612

Time Allowed : Three (3) Hours

Instructions

1. This paper consists of TWO sections.
   a. SECTION A(COMPULSORY): 60 MARKS
      Answer ALL QUESTIONS.
   b. SECTION B: 40 MARKS
      Answer ANY TWO questions.

2. Show all your working.
3. Special requirements: None

THIS PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GIVEN BY THE INVIGILATOR.

Section A: Answer ALL Questions

QUESTION 1

Find **second order** perturbation approximate solutions for the following system of equations.

$$\varepsilon x + 10y = 21$$
$$5x + y = 7.$$

Solve approximately for $\varepsilon = 0.01$. [20]

QUESTION 2

Use regular perturbation to find the **third order** approximate solution for the initial value problem;

$$\frac{df(x)}{dx} + f(x) - \varepsilon f^2(x) = 0, \quad f(0) = 2.$$

[20]

QUESTION 3

Consider the following second order boundary value equation;

$$\varepsilon y'' + y' = 1 + 2x, \quad y(0) = 0, \quad y(1) = 1.$$

Assume that the boundary layer occurs at $x = 0$. Use the method of Dominant Balance to find an approximate solution $y(x, \varepsilon)$ for the equation. [20]

Section B: Answer ANY 2 Questions

QUESTION 4

a. Use the exponential substitution $y = e^{u(x)/\varepsilon}$ to construct a WKB approximation to the differential equation $\varepsilon^2 y'' - k(x)^2 y = 0$, where k(x) is a coefficient function, independent of the small parameter ε. [15]

b. Use your result in a. above to construct a WKB approximation for the differential equation; $\varepsilon^2 y'' - (1+x)^2 y = 0, \quad x \geq 0.$ [5]

QUESTION 5

Apply the Homotopy Perturbation Method to the following system of nonlinear ordinary differential equations. Formulate the equations and corresponding boundary conditions to be solved for a **third order** approximation.

$$f'''(\eta) + \frac{1}{2} f(\eta) f''(\eta) = 0, \quad f(0) = 0, \quad f'(0) = 0, \quad f'(\infty) = 1,$$
$$\varepsilon \theta''(\eta) + \frac{1}{2} f(\eta) \theta'(\eta) = 0, \quad \theta(0) = 1, \quad \theta(\infty) = 0.$$

where $\varepsilon = \dfrac{1}{Pr}$. Show that

$$f_0 = \frac{1}{10}\eta^2, \quad f_1 = -\frac{1}{6000}\eta^5 + \frac{5}{96}\eta^2.$$
$$\theta_0 = -\frac{1}{5}\eta + 1, \quad \theta_1 = \frac{Pr}{1200}\eta^4 - \frac{5Pr}{48}\eta.$$

[20]

QUESTION 6

Consider the following problem and answer the questions that follow:

A firm is interested in understanding the dynamics of computer malware in one of it's laboratories. A mathematical analyst decided to use mathematical modeling to understand the dynamics of computer malware in the laboratory. The analyst assumed that the computer malware model consists of six compartments, namely, Immune node, M, susceptible node, S, exposed node, E, infectious node, I, quarantine node, Q, and recovered node, R. Dynamical transfer assumptions are well governed by non-linear first order ordinary differential equations where β, π, μ and λ are positive constants and ϕ, η, ξ, α, b, ϵ, σ, γ, δ, and ρ are non-negative constants. The constant π is the recruitment rate of susceptible class S to the computer network and μ is the per capita natural mortality rate (that is, the removal rate of the nodes from the network for reasons different from the attack of malicious objects). The constant β is the recruitment rate of the immune class M, alongside with it loss of immunity rate η to be become susceptible. The constant ϵ is rate for nodes leaving the exposed compartments E for infective class I, α is the probability of susceptible class either becoming exposed first or infectious immediately (right away), δ is the constant removal rate from the network of nodes I and Q due to attack of malicious objects, σ and ρ are the constant rates at which the nodes I and Q transient recovery after the run of anti-malicious software and the classes are immediately transferred to the recovery compartment, while γ is the constant rate at which the infective are remove from node I to quarantine compartment and ϕ is the constant rate gain of immunity whereas ξ is the rate of recovered class R to become susceptible to the malware. The constant λ is the rate of force of infection.

1. Write down the flow model [10]
2. Write down the nonlinear first order ordinary differential equation model [10]

END OF EXAMINATION