UNIVERSITY OF ESWATINI

81

SUPPLEMENTARY/RE-SIT EXAMINATION 2020

BSc IV, B.Ed IV, BASS IV

Title of Paper : Numerical Analysis II

Course Number : MAT411/M411

Time Allowed : Three (3) Hours

Instructions

1. This paper consists of SIX (6) questions in TWO sections.
2. Section A is **COMPULSORY** and is worth 40%. Answer ALL questions in this section.
3. Section B consists of FIVE questions, each worth 20%. Answer ANY THREE (3) questions in this section.
4. Show all your working.
5. Start each new major question (A1, B2 – B6) on a new page and clearly indicate the question number at the top of the page.
6. You can answer questions in any order.
7. Indicate your program next to your student ID.

Special Requirements: NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GIVEN BY THE INVIGILATOR.

82

SECTION A [40 Marks]: ANSWER ALL QUESTIONS

QUESTION A1 [40 Marks]

A1 (a) Determine if the differential equation

$$y'(x) = (2x-3)y(x) + 1, \quad 1 \le x \le 3, \;\; y(1) = 1$$

has a unique solution for $1 \le x \le 3$. [5 Marks]

(b) Use the Runge-Kutta method (RK4) to solve the IVP

$$y' = 2ty, \quad y(0) = 2$$

for $0 \le t \le 0.2$ with $h = 0.1$ and compare the approximate solution against the exact solution $y(t) = 2e^{t^2}$ [9 Marks]

(c) Consider the multi-step method defined by the scheme

$$y_{i+1} + y_i - 2y_{i-1} = \frac{h}{2}(f_{i-1} + 5f_i)$$

Discuss the consistency, stability and convergence of this scheme. [8 Marks]

(d) Show that the linear least squares approximation of $f(x) = \sqrt{x}$ on the interval $[0,1]$ is $P_1(x) = \frac{4}{15} + \frac{4}{5}x$ [6 Marks]

(e) What are the two conditions that are necessary for an initial value problem

$$y'(t) = f(t, y), \quad a \le t \le b, \;\; y(a) = \alpha$$

to be well-posed? [2 Marks]

(f) Derive the Euler method for solving initial value problems of the form

$$y'(t) = f(t, y(t)), \quad y(t_0) = y_0$$

where y_0 is a given initial condition [4 Marks]

(g) Compute the local truncation error of the 2-step Adams-Moulton method [6 Marks]

$$y_{i+1} = y_i + \frac{h}{12}(5f(t_{i+1}, y_{i+1}) + 8f(t_i, y_i) - f(t_{i-1}, y_{i-1}))$$

83

SECTION B: ANSWER ANY *THREE* QUESTIONS

QUESTION B2 [*20 Marks*]

B2 (a) Suppose that $w(x)$ is a weight function and $\{\phi_0, \phi_1, \ldots, \phi_n\}$ is a set of linearly independent functions on $[a, b]$.
If the set $\{\phi_0, \phi_1, \ldots, \phi_n\}$ is orthogonal, show that the coefficients of the polynomial $P_n(x) = \sum_{k=0}^{n} c_k\phi_k(x)$ that minimizes the error

$$E(c_0, c_1, \ldots, c_n) = \int_a^b w(x)\left(f(x) - \sum_{k=0}^{n} a_k\phi_k(x)\right)^2 dx \qquad (1)$$

are

$$c_j = \frac{1}{\alpha_j}\int_a^b w(x)f(x)\phi_j(x)\ dx, \qquad (2)$$

where $\alpha_j = \int_a^b w(x)[\phi_j(x)]^2$ [6 Marks]

(b) Find the least squares quadratic function of the form $ax^2 + b$, which best fits the curve $y = \sqrt{2x+1}$ over the interval $0 \leq x \leq \frac{3}{2}$. [14 Marks]

QUESTION B3 [*20 Marks*]

B3 (a) Derive the backward in time and central in space (BTCS) implicit finite difference scheme for solving the heat equation [3 Marks]

$$u_t = u_{xx}$$

(b) Use the Von-Neumann analysis to show that the implicit finite difference scheme for solving the heat equation is unconditionally stable. [8 Marks]

(c) Use an $O(h^2)$ finite difference scheme to solve the following boundary-value problem using a step size $h = 1/3$ and compare the results against the exact solution $U(x) = -x^2 + 3x + 2$. [9 Marks]

$$U''(x) + 3xU'(x) - 3U(x) = -3x^2 - 8, \quad U(0) = 2, \ U(1) = 4$$

QUESTION B4 [*20 Marks*]

B4 (a) Use the method of undetermined coefficients to derive the 2-Step Adams-Moulton (implicit) method. [10 Marks]

(b) Consider the initial value problem

$$x' = 2x + 3y + 5z$$
$$y' = 2x + y - 5z$$
$$z' = -5x - 5y + 3z$$

84

with initial conditions $x(0) = -2,\ y(0) = 1,\ z(0) = 0$ and exact solutions

$$x(t) = -\frac{3}{20}e^{-t} - \frac{25}{4}e^{3t} + \frac{22}{5}e^{4t}$$
$$y(t) = \frac{3}{20}e^{-t} + \frac{25}{4}e^{3t} - \frac{27}{5}e^{4t}$$
$$z(t) = -5e^{3t} + 5e^{4t}$$

Use the Euler's method with $h = 0.1$ to find the solution of $x(0.2), y(0.2), z(0.2)$ and compute the error at $t = 0.2$.. [10 Marks]

QUESTION B5 [*20 Marks*]

B5 Apply finite differences with forward-difference approximation in time to solve the following PDE, subject to the given initial and boundary conditions

$$U_t - U_{xx} = \frac{tx}{4} + 1, \quad 0 < x < 1,\ t > 0$$
$$U(0,t) = 0.7, \quad U(1,t) = 0.8$$
$$U(x,0) = -4x^2 + x + 4$$

Assume that the step sizes in x and t are $h = 0.2$ and $k = 0.1$ respectively, and compute the approximate solutions at $t = 0.1$ and $t = 0.2$. [20 Marks]

85

QUESTION B6 [*20 Marks*]

B6 Let $f(x)$ be a function defined as

$$f(x) = \begin{cases} x + \pi\ , & -\pi < x < 0 \\ \pi - x\ , & 0 < x < \pi. \end{cases}$$

(a) Show that the least squares trigonometric polynomial that approximates $f(x)$ in the interval $-\pi < x < \pi$ is

$$\begin{aligned} S_n(x) &= \frac{\pi}{2} - \sum_{k=1}^{n} \frac{2\left[(-1)^k - 1\right]}{\pi k^2} \cos kx \\ &= \frac{\pi}{2} + \frac{4}{\pi}\left[1 + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \frac{\cos 7x}{7^2} + \frac{\cos 9x}{9^2} + \dots\right] \end{aligned}$$

for $n = 1, 2, \dots$ [16 Marks]

(b) By giving an appropriate value to x, show that as $n \to \infty$

$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \dots$$

[4 Marks]

END OF EXAMINATION PAPER