UNIVERSITY OF ESWATINI

MAIN EXAMINATION, 2019/2020

BASS III, B.Ed (Sec.) III, B.Sc. III, B.Eng. III

Title of Paper : Complex Analysis

Course Number : MAT313

Time Allowed : Three (3) Hours

Instructions

1. This paper consists of SIX (6) questions in TWO sections.
2. Section A is **COMPULSORY** and is worth 40%. Answer ALL questions in this section.
3. Section B consists of FIVE questions, each worth 20%. Answer ANY THREE (3) questions in this section.
4. Show all your working.
5. Start each new major question (A1, B2 – B6) on a new page and clearly indicate the question number at the top of the page.
6. You can answer questions in any order.
7. Indicate your program next to your student ID.

Special Requirements: NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GIVEN BY THE INVIGILATOR.

SECTION A [*40 Marks*]: ANSWER ALL QUESTIONS

QUESTION A1 [*40 Marks*]

a) Evaluate $\sin^{-1}(1)$ and leave your answer in the form $a+ib$. [5]

b) Find real constants a and b so that the function

$$f(z) = 3x - y + 5 + i(ax + by - 3)$$

is analytic. [5]

c) i) Evaluate $\int_C \frac{e^{\pi z} - 3\sin^2(z) + 4z^3 z^i}{z^2 - 7z + 12} dz$ where C is given by $|z| = 2$. [5]

ii) State Morera's theorem. [5]

d) Find the Maclaurin series of $\Phi(z) = z^4 \sin(2z)$. [5]

e) Express $\int_0^{2\pi} \frac{d\theta}{5 + 4\sin(\theta)}$ as a contour integral around the unit circle $|z| = 1$. [5]

f) Find the value of the residue at $z = 3$ for $f(z) = \frac{2z}{(z-4)^2(z-3)}$. [5]

g) Express $z = \frac{(1 - i\sqrt{3})^5}{(i-1)^{10}}$ in the form $z = a + ib$. [5]

SECTION B: ANSWER ANY *THREE* QUESTIONS

QUESTION B2 [*20 Marks*]

a) Consider the function $f(z) = \frac{1}{(z-1)^2(z-3)}$.

i) Locate and classify all singularities. [2]

ii) Find the value of the residue at each singularity. [6]

iii) Hence evaluate $\int_C \frac{4}{(z-1)^2(z-3)} dz$, where C is the rectangle defined by $x = 0$, $x = 4$, $y = -1$, $y = 1$. [2]

b) Using Cauchy's Residue Theorem, evaluate

$$\int_0^{2\pi} \frac{\cos(\theta) d\theta}{5 - 4\cos(\theta)}.$$

[10]

QUESTION B3 [*20 Marks*]

a) Prove that when a limit of a function $\phi(z)$ exists at a point z_0, it is unique. [8]

b) Consider the function $\alpha(x, y) = (x - y)(x + y)$.

i) Show that $\alpha(x, y)$ is harmonic. [3]

ii) Find the harmonic conjugate $\beta(x, y)$. [7]

iii) Hence or otherwise, find the analytic function $w(z) = \alpha(x, y) + i\beta(x, y)$ such that $w(i) = 1$. [2]

QUESTION B4 [*20 Marks*]

a) Evaluate $\int_C (3\overline{z} - 2z)dz$ where C is parametrized by $z = it^2$ for $t \in [0, 1]$. [7]

b) Evaluate $\int_C \frac{5z + 7}{z^2 + 2z - 3}dz$ where C is given by $|z - 2| = 2$. [5]

c) Evaluate $\int_C \frac{z + 1}{z^4 + 2iz^3}dz$ where C is parametrized by $z = e^{i\theta}$ for $\theta \in [0, 2\pi]$. [8]

QUESTION B5 [*20 Marks*]

a) Determine if the sequence $\left\{\frac{3ni + 2}{n(1 + i)}\right\}$ for $n = 1, 2, \cdots$ converges or diverges. [4]

b) Find the Maclaurin series that represents the function $\Gamma(z) = z^3 \cos(3z)$. [6]

c) Find the Laurent series that represents the function $f(z) = \frac{1}{(z - 2)(z - 3)}$ in the domain

$$2 < |z| < 3.$$

[10]

QUESTION B6 [*20 Marks*]

a) i) Show that $\tanh^{-1}(z) = \frac{1}{2}\ln\left(\frac{1 + z}{1 - z}\right)$ [10]

ii) Hence show that $\frac{d}{dz}\left(\tanh^{-1}(z)\right) = \frac{1}{1 - z^2}$ [4]

b) Find the principal value of $z = i^{\frac{i}{\pi}}$ [6]

END OF EXAMINATION PAPER