UNIVERSITY OF ESWATINI 59

SUPPLEMENTARY/RE-SIT EXAMINATION 2020

BSc III, B.Ed III, BASS III, BEng IV

Title of Paper : Numerical Analysis I

Course Number : MAT311/M311

Time Allowed : Three (3) Hours

Instructions

1. This paper consists of SIX (6) questions in TWO sections.
2. Section A is **COMPULSORY** and is worth 40%. Answer ALL questions in this section.
3. Section B consists of FIVE questions, each worth 20%. Answer ANY THREE (3) questions in this section.
4. Show all your working.
5. Start each new major question (A1, B2 – B6) on a new page and clearly indicate the question number at the top of the page.
6. You can answer questions in any order.
7. Indicate your program next to your student ID.

Special Requirements: NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GIVEN BY THE INVIGILATOR.

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SECTION A [*40 Marks*]: ANSWER ALL QUESTIONS

QUESTION A1 [*40 Marks*]

A1 (a) Suppose you need to evaluate $f(x) = \sqrt{x^4 + 4} - 2$ for x near 0.

i. Show that a direct calculation of $f(0.5)$ using the definition of $f(x)$ with 3-digit rounding arithmetic can lead to large relative errors. Why is this? [3 Marks]

ii. Derive an alternative formula for $f(x)$ that has better round-off error properties. Illustrate by using your formula to calculate $f(0.5)$ and the corresponding relative error. [5 Marks]

(b) i. Show that $e^x + x^2 - 5 = 0$ has exactly one root in the interval $[0, 2]$ [5 Marks]

ii. How many steps of the bisection method are required to approximate the root to within 10^{-5} [4 Marks]

(c) Find the decimal equivalent of the following single precision machine number.

1	10000011	10101001100000000000000

[7 Marks]

(d) Let $f(x) = \sqrt{x}$. Compute the second degree Lagrange interpolating polynomial, $P_2(x)$, for $f(x)$ using the points $x_1 = 1$, $x_2 = \frac{9}{4}$ and $x_3 = 4$ [5 Marks]

(e) Use Newton's divided differences to find a polynomial that passes through the points

$$(-1, 0),\ (2, 1),\ (3, 1),\ (5, 2).$$

[6 Marks]

(f) Use the composite Trapezoid rule with $n = 3$ to compute

$$\int_0^3 \frac{2}{x^2 + 4}\, dx$$

[5 Marks]

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SECTION B: ANSWER ANY *THREE* QUESTIONS

QUESTION B2 [*20 Marks*]

B2 (a) Find the interpolating polynomial passing through the three points

$$(-2, 33), \quad (2, 29), \quad (5, 236)$$

using the Vandermonde matrix approach. [10 Marks]

(b) Construct a Newton's forward difference table corresponding to the following data

x	-5	-3	-1	1	3
$f(x)$	25	9	1	2	13

and show that the polynomial of least degree that goes through the points is

$$25 - 8(x+5) + (x+3)(x+5) + \frac{1}{48}(x+1)(x+3)(x+5)$$

[10 Marks]

QUESTION B3 [*20 Marks*]

B3 (a) Show that the sequence defined by

$$x_{n+1} = \ln(2x_n + 1)$$

converges to the exact solution of

$$e^x - 2x - 1 = 0$$

for any starting value $x_0 \in [1, 2]$. [10 Marks]

(b) Let $f(x) = 4x + 2\sqrt{x} - 5$.

i. Show that the Newton method scheme for solving $f(x) = 0$ is

$$x_{n+1} = \frac{5\sqrt{x_n} - x_n}{4\sqrt{x_n} + 1}$$

[5 Marks]

ii. Starting from $x_0 = 0.5$, find the first four iterations that give an approximation of the solution [5 Marks]

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QUESTION B4 [20 Marks]

B4 Find the approximation solution for the linear system

$$\begin{aligned} x_1 &+ 2x_2 + 4x_3 = 3 \\ -2x_1 &+ x_3 = -1 \\ x_1 &- 3x_2 + x_3 = 5 \end{aligned}$$

using

(a) two iterations of the Jacobi Method with starting point (0,0,0) [4 Marks]

(b) two iterations of the Gauss-Siedel Method with starting point (0,0,0) [4 Marks]

(c) the Doolittle LU factorisation method. [12 Marks]

QUESTION B5 [20 Marks]

B5 (a) Construct a quadrature rule on the interval $[1, 5]$ using the nodes 2, 3, 4. [8 Marks]

(b) Use the quadrature rule derived in (a) above to estimate the integral

$$\int_1^5 \frac{2}{x^2+4}\,dx$$

[4 Marks]

(c) Use the Simpson's method to estimate the integral $\int_0^1 e^{1-x^2}$ using a step size of $h = 0.25$. [5 Marks]

(d) Estimate the error of the result in (c) above [3 Marks]

QUESTION B6 [20 Marks]

B6 (a) Use Taylor series to derive the second backward finite difference and second centred difference methods methods given by

$$f'(x_i) \approx \frac{3f(x_i) - 4f(x_{i-1}) + f(x_{i-2})}{2h}$$

and

$$f'(x_i) \approx \frac{-f(x_{i+2}) + 8f(x_{i+1}) - 8f(x_{i-1}) + f(x_{i-2})}{12h}$$

respectively. [10 Marks]

(b) Use the two formulas in (a) with $h = 0.25$ to approximate $f'(0.5)$ when $f(x) = e^x$. [7 Marks]

(c) Compute the relative error of each approximation [3 Marks]