UNIVERSITY OF ESWATINI

DECEMBER 2019 MAIN EXAMINATION

BSc III, B.Ed III, BASS III, BEng IV

Title of Paper : Numerical Analysis I

Course Number : MAT311/M311

Time Allowed : Three (3) Hours

Instructions

1. This paper consists of SIX (6) questions in TWO sections.
2. Section A is **COMPULSORY** and is worth 40%. Answer ALL questions in this section.
3. Section B consists of FIVE questions, each worth 20%. Answer ANY THREE (3) questions in this section.
4. Show all your working.
5. Start each new major question (A1, B2 – B6) on a new page and clearly indicate the question number at the top of the page.
6. You can answer questions in any order.
7. Indicate your program next to your student ID.

Special Requirements: NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GIVEN BY THE INVIGILATOR.

SECTION A [*40 Marks*]: ANSWER ALL QUESTIONS

QUESTION A1 [*40 Marks*]

A1 (a) Convert $\frac{1917}{32}$ to binary form. [5 Marks]

(b) Convert the binary number 100010.110111 to decimal format. [4 Marks]

(c) Use the intermediate value theorem to check if the function $f(x) = \sin(x)$ has a solution in the interval [1,2]? [3 Marks]

(d) Solve the following linear system of equations

$$\begin{aligned} -x_1 + 2x_2 + 6x_3 &= -12, \\ x_1 - 7x_2 - 3x_3 &= 11, \\ x_1 + 18x_2 - 17x_3 &= 14, \end{aligned}$$

given that the LU decomposition of the coefficient matrix is

$$L = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & -4 & 1 \end{pmatrix} \text{ and } U = \begin{pmatrix} -1 & 2 & 6 \\ 0 & -5 & 3 \\ 0 & 0 & 1 \end{pmatrix}$$

[7 Marks]

(e) Use the quadrature formula

$$\int_0^2 f(x)dx = \frac{1}{3}f(0) + \frac{4}{3}f(1) + \frac{1}{3}f(2)$$

to compute the integral

$$\int_0^2 \frac{1}{x^2+4}dx.$$

[3 Marks]

(f) Find the polynomial that interpolates the given data

x	0	1	2
$f(x)$	1	1	4

using the Lagrange interpolation. [7 Marks]

(g) How can accurate values of the function

$$f(x) = x - \sin x$$

be computed near $x = 0$? [5 Marks]

(h) Starting with $x_0 = 1.5$, perform three iterations to find an approximate root of the equation $x^3 - x - 1 = 0$, using the Newton's method [6 Marks]

SECTION B: ANSWER ANY *THREE* QUESTIONS

QUESTION B2 [*20 Marks*]

B2 (a) List the four key conditions that must be satisfied in the fixed point theorem for a given function $g(x)$ to have a fixed point in an interval $[a, b]$ [4 Marks]

(b) Show that $g(x) = \frac{3^{-x}}{4}$, which can be derived from the nonlinear function $f(x) = 3^{-x} - 4x = 0$, gives a sequence $x_{n+1} = g(x_n)$ that converges to a unique fixed point in $[0, 1]$ [8 Marks]

(c) An approximate solution of the equation $2x^3 - 7x + 5 = 0$ can be obtained from the fixed point iteration scheme

$$x_{n+1} = \frac{1}{7}(5 + 2x_n^3)$$

with $x_0 = 0.5$ as the initial guess in $[0, 1]$. Starting from the given x_0 find the number of iterations that are required to estimate the solution to within 0.000001 [8 Marks]

QUESTION B3 [*20 Marks*]

B3 (a) Determine the decimal number that has the following single precision representation [7 Marks]

1	1000101	01111001001111010000000

(b) Derive the Jacobi iteration scheme for the following linear system and use 2 iterations to approximate the solution of

$$\begin{aligned} 15x_1 - 2x_2 - 8x_3 &= 38 \\ -2x_1 - 6x_2 + 3x_3 &= -21 \\ -7x_1 + 7x_2 + 16x_3 &= -73 \end{aligned}$$

starting from $(0, 0, 0)$ as initial approximation. [7 Marks]

(c) i. Prove that the function $2^x - 5x = 0$ has a solution in $[0, 1]$. [2 Marks]

ii. If the root of $2^x - 5x = 0$ exists in $[0, 1]$, use 4 iterations of the Bisection method to approximate the root. [4 Marks]

QUESTION B4 [*20 Marks*]

B4 (a) Evaluate the integral $\int_0^2 \ln(1+x)\,dx$ by the trapezoid rule with an accuracy of at least $\varepsilon = 0.05$ [10 marks]

(b) The quadrature formula

$$\int_{-1}^{1} f(x)\,dx \approx c_0 f(-1) + c_1 f(0) + c_2 f(1)$$

is exact for all polynomials of degree less than or equal to 2. Determine c_0, c_1 and c_2. [5 marks]

(c) Find the constants c_0, c_1 and x_1 so that the quadrature formula

$$\int_0^1 f(x)dx = c_0 f(0) + c_1 f(x_1)$$

has the highest possible degree of precision. [5 marks]

QUESTION B5 [*20 Marks*]

B5 (a) Construct a Newton's forward difference table corresponding to the following data and find a polynomial of least degree that goes through the points. [10 Marks]

x	2	4	6	8
$f(x)$	6	23	56	118

(b) Use Lagrange functions to construct a quadrature rule on the interval $[-2, 2]$ using the nodes -2, 0, 2. [10 Marks]

QUESTION B6 [*20 Marks*]

B6 (a) Find the LU factorisation of the matrix A in which U is a unit upper triangular matrix and L is a lower triangular matrix (Crout Method). [10 Marks]

$$A = \begin{pmatrix} -1 & -6 & -2 \\ 2 & 10 & 0 \\ -4 & -20 & 5 \end{pmatrix}$$

(b) i. Derive the approximation formula

$$f'(x) \approx \frac{1}{2h}[4f(x+h) - 3f(x) - f(x+2h)]$$

and show that its error term is $\frac{h^2}{3} f'''(\xi)$ [6 Marks]

ii. Use the formula in (i) above to approximate $f'(1.9)$ with $f(x) = ln(x)$ using $h = 0.1, 0.01$. Compute the error in each case. [4 marks]

END OF EXAMINATION PAPER