University of Eswatini

Main Examination, 2018/2019

BASS IV, B.Sc IV

Title of Paper : INTRODUCTION TO MATHEMATICS OF FINANCE

Course Number : MAT 442

Date : 04 June 2019

Time Allowed : Three (3) Hours

Instructions:

1. This paper consists of SIX (6) questions in TWO sections.
2. Section A is **COMPULSORY** and is worth 40%. Answer ALL questions in this section.
3. Section B consists of FIVE questions, each worth 20%. Answer ANY THREE (3) questions in this section.
4. Start each new major question (A1-A5, B1 - B5) on a new page and clearly indicate the question number at the top of the page.
5. You can answer questions in any order.
6. Indicate your program next to your student ID.

Special Requirements: NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GIVEN BY THE INVIGILATOR.

SECTION A [*40 Marks*]: ANSWER ALL QUESTIONS

QUESTION A1.

a.) Define a random variable X. [2 marks]

b.) Suppose $X : \Omega \to \Re$ is a random variable that assumes countably many values $a_1, a_2, a_3, ... \in \Re$. Show that X is a random variable if $X^{-1}(a_k) \in \Im \quad \forall\, k \in \mathbf{N}$. [6 marks]

QUESTION A2.

a.) Define a stochastic process $\{X_t\}_{t \geq 0}$. [2 marks]

b.) Let $X : \Omega \to \Re$ be a random variable with cumulative distribution function $F(x)$ given by $F(x) = P[X \leq x]$. Show that F is non decreasing and right continuous in $\Re$. [6 marks]

QUESTION A3.

a.) Define a σ-algebra. [4 marks]

b.) Given a probability space $(\Omega, \Im, P)$ and a random variable X. When is X said to be $\Im$-measurable. [4 marks]

QUESTION A4.

a.) Define a Brownian Motion and give two (2) financial industry-based Brownian processes. [4 marks]

b.) Given a Brownian price process $B(t)$. Show that the increments $B_{t_1}, (B_{t_2} - B_{t_1}), (B_{t_3} - B_{t_2}), ...$ at distinct times $t_i; i = 1, 2, 3, ...$ are independent. [4 marks]

QUESTION A5.

a.) Given a market process $X(t)$ whose changes in time is described by

$$\frac{dX(t)}{dt} = \alpha X(t) + \rho X(t).''noise.''$$

If noise $\sim W(t)$, give the basic properties of $W(t)$. [3 marks]

b.) Suppose $W(t) \sim B(t)$. Construct a general solution for $X(t)$. [5 marks]

SECTION B: ANSWER ANY *THREE* QUESTIONS

QUESTION B1

a.) Define an elementary function. [2 marks]

b.) List 3 real valued functions that are elementary functions. [3 marks]

c.) State and prove Ito isometry for elementary and bounded function $\phi(t, \omega)$. [15 marks]

QUESTION B2.

a.) Define an Ito Integral. [3 marks]

b.) List four (4) properties of the Ito integral. [4 marks]

c.) Evaluate the integral $I = \int_0^t sB^3(s)dB(s)$. [13 marks]

QUESTION B3.

a.) Define an arbitrage market. [3 marks]

b.) Consider the market process $X(t)$ given by

$$dX_1(t) = 2dt + dB_1(t), dX_2(t) = -dt + dB_1(t) + dB_2(t).$$

If $dX_0(t) = 0$, show that a portfolio $\theta(t) = (\theta_0, 1, 1)$ should be allowed to do business in $X(t)$. [13 marks]

c.) Find the value process $V^\theta(t)$ at expiration time $t = T$. [4 marks]

QUESTION B4.

a.) Define a martingale. [3 marks]

b.) Use Ito's formula to prove that if

$$dZ(t) = Z(t)\theta(t, \omega)dB(t)$$

then $Z(t)$ is a martingale for all $t \leq T$ provided that $Z(t)\theta_k(t, \omega) \in \nu(0, T) \quad 1 \leq k \leq n$. [17 marks]

QUESTION B5.

a.) Use Ito's formula to write $X(t)$ in the form

$$dX(t) = u(t, \omega)dt + v(t, \omega)dB(t)$$

(i.) $X(t) = 4B^2(t)$. [4 marks]

(ii.) $X(t) = 2 + t + e^{B(t)}$. [4 marks]

b.) Find the solution to the Black-Scholes equation

$$dS(t) = 2S(t)dt + 0.2S(t)dB(t); \quad S(0) = 10$$

representing the change in price $S(t)$ of an option trading in an African stock market at time $t \in [0, t]$. [12 marks]

END OF EXAMINATION