UNIVERSITY OF ESWATINI

MAIN EXAMINATION, 2018/2019

BSc.IV, BASS IV, BEd IV

Title of Paper : Mathematical Statistics II

Course Number : MAT441

Time Allowed : Three (3) Hours

Instructions

1. This paper consists of SIX (6) questions in TWO sections.
2. Section A is **COMPULSORY** and is worth 40%. Answer ALL questions in this section.
3. Section B consists of FIVE questions, each worth 20%. Answer ANY THREE (3) questions in this section.
4. Show all your working.
5. Start each new major question (A1, B2, B3 ,B4, B5, B6) on a new page and clearly indicate the question number at the top of the page.
6. You can answer questions in any order.
7. Indicate your program next to your student ID.

Special Requirements: NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GIVEN BY THE INVIGILATOR.

SECTION A [*40 Marks*]: ANSWER ALL QUESTIONS

QUESTION A1 [*40 Marks*]

A1 (a) Suppose that $X_1, X_2, \ldots X_n$ and $Y_1, Y_2, \ldots Y_n$ are independent random samples from populations with means μ_1 and μ_2 and variances σ_1^2 and σ_2^2, respectively. Show that the random variable

$$U_n = \frac{(\overline{X} - \overline{Y}) - (\mu_1 - \mu_2)}{\sqrt{(\sigma_1^2 + \sigma_2^2)/n}}$$

satisfies the conditions of the Central Limit theorem and that the distribution function of U_n converges to a standard normal distribution function as $n \longrightarrow \infty$.

[5 Marks]

(b) $X_1, X_2, \ldots, X_n$ is a random sample from the uniform distribution between θ and 1 (i.e. $f(x) = (1-\theta)^{-1}$ for $\theta < x < 1$), where $\theta (< 1)$ is an unknown parameter. Denote the sample mean by $\overline{X}$. Show that the method of moments estimator, $\hat{\theta}$, of θ is

$$2\overline{X} - 1.$$

[5 Marks]

(c) Let $X_1, \ldots, X_n$, $n > 4$, be a random sample from a population with a mean μ and variance σ^2. Consider the following two estimators of μ:

$$\hat{\theta}_1 = \frac{1}{9}(X_1 + 2X_2 + 5X_3 + X_4),$$

$$\hat{\theta}_2 = \overline{X}.$$

Calculate the relative efficiency $e(\hat{\theta}_2, \hat{\theta}_1)$. Interpret.

[5 Marks]

(d) A large-sample $\alpha-$level test of hypothesis for $H_0 : \theta = \theta_0$ versus $H_a : \theta > \theta_0$ rejects the null hypothesis if

$$\frac{\hat{\theta} - \theta_0}{\sigma_{\hat{\theta}}} > z_\alpha.$$

Show that this is equivalent to rejecting H_0 if θ_0 is less than the large-sample $100(1-\alpha)\%$ lower confidence bound for θ.

[5 Marks]

(e) Write down the four elements of a statistical test.

[4 Marks]

(f) Auditors are often required to compare the audited (or current) value of an inventory item with the book (or listed) value. If a company is keeping its inventory and books up to date, there should be a strong linear relationship between the audited and book values. A company sampled ten inventory items and obtained the audited and book values given in the accompanying table.

Item	Audit Value (y_i)	Book Value (x_i)
1	9	10
2	14	12
3	7	9
4	29	27
5	45	47
6	109	112
7	40	36
8	238	241
9	60	59
10	170	167

Fit the model $Y = \beta_0 + \beta_1 x + \varepsilon$ to these data.

[10 Marks]

(g) Let X be a binomial random variable with parameters n and p. Assume that the prior distribution of p is uniform on $[0, 1]$. Find the posterior distribution, $f(p|x)$.

[6 Marks]

SECTION B: ANSWER ANY *THREE* QUESTIONS

QUESTION B2 [*20 Marks*]

B2 (a) A study of parallel interchange ramps revealed that many drivers do not use the entire length of parallel lanes for acceleration, but seek, as soon as possible, a gap in the major stream of traffic to merge. At one site on the highway, 46% of drivers used less than one third of the lane length available before merging. Suppose we monitor the merging pattern of a random sample of 250 drivers at this site.

(i) What is the probability that fewer than 120 of the drivers will use less than one third of the acceleration lane length before merging?

[3 Marks]

(ii) What is the probability that more than 225 of the drivers will use less than one third of the acceleration lane length before merging?

[3 Marks]

(b) Let $X_1, \ldots, X_n$ be independent and identically distributed random variables with variance $\sigma^2 < \infty$. If

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X})^2$$

is the variance of a random sample from an infinite population, show that S^2 is an unbiased estimator for σ^2.

[8 Marks]

(c) The reaction of an individual to a stimulus in a psychological experiment may take one of two forms, A or B. If an experimenter wishes to estimate the probability p that a person will react in manner A, how many people must be included in the experiment? Assume that the experimenter will be satisfied if the error of estimation is less than 0.04 with probability equal to 0.90. Assume also that he expects p to lie somewhere in the neighborhood of 0.6.

[6 Marks]

QUESTION B3 [*20 Marks*]

B3 (a) Let $X_1, \ldots, X_n$ be a random sample from a population with pdf

$$f(x) = \begin{cases} \frac{1}{\alpha}x^{(1-\alpha)/\alpha}, & \text{for } 0 < x < 1; \alpha > 0 \\ 0, & \text{otherwise} \end{cases}$$

Show that the maximum likelihood estimator of α is $\hat{\alpha} = -(1/n)\sum_{i=1}^{n}\ln(X_i)$. [5 Marks]

(b) Now suppose $f(x,\theta) = \frac{1}{\theta}e^{-x/\theta}, x > 0$, and $\overline{X} = \frac{1}{n}\sum_{i=1}^{n}X_i$.

i. Show that $\overline{X}$ is an unbiased estimator of θ. [5 Marks]

ii. State the factorisation criterion for sufficient statistics and use it to show that $\overline{X}$ is sufficient for θ. [5 Marks]

iii. State the Cramer-Rao inequality for unbiased estimators of θ. Show that $\overline{X}$ attains the lower bound for the distribution above. Explain what that means regarding the efficiency of $\overline{X}$ as an estimator of θ. [5 Marks]

QUESTION B4 [*20 Marks*]

B4 (a) In the data set below, W denotes the weight (in pounds) and l the length (in inches) for 15 alligators captured in central Florida. Because l is easier to observe than W for alligators in their natural habitat, a researcher would like to construct a model relating weight to length. Such a model can then be used to predict the weights of alligators of specified lengths.

Alligator	length(l)	Weight (W)
1	47.94	130.32
2	36.97	50.91
3	75.94	639.06
4	30.88	27.94
5	45.15	79.84
6	46.06	109.95
7	31.82	33.12
8	42.94	90.12
9	33.12	35.87
10	35.87	38.09
11	66.02	365.04
12	43.82	83.93
13	40.85	79.84
14	41.68	83.10
15	43.82	70.12

Fit the model $E(W) = \alpha_0 l^{\alpha_1}$.

[10 Marks]

(b) Given the data

x	y
-2	0
-1	0
0	1
1	1
2	3

and

$$(XY)^{-1} = \begin{bmatrix} 17/35 & 0 & -1/7 \\ 0 & 1/10 & 0 \\ -1/7 & 0 & 1/14 \end{bmatrix},$$

Fit the model $Y = \beta_0 + \beta_1 x + \beta_2 x^2 + \varepsilon$.

[10 Marks]

QUESTION B5 [*20 Marks*]

B5 (a) A psychological study was conducted to compare the reaction times of men and women to a stimulus. Independent random samples of 50 men and 50 women were employed in the experiment.The results are shown in Table below.

Men	Women
$n_1 = 50$	$n_2 = 50$
$\overline{y}_1 = 3.6$ seconds	$\overline{y}_2 = 3.8$ seconds
$s_1^2 = 0.18$	$s_2 = 0.14$

(i) Do the data present sufficient evidence to suggest a difference between true mean reaction times for men and women? Use $\alpha = 0.05$.

[6 Marks]

(ii) Find the p-value for the statistical test.

[2 Marks]

(b) State the Neyman-Pearson lemma.

[4 Marks]

(c) A random sample of size 36 from a population with known variance, $\sigma^2 = 9$, yields a sample mean of $\overline{x} = 17$. Compute the type II error β for testing the hypothesis $H_0 : \mu = 15$ versus $H_a : \mu = 16$. Assume $\alpha = 0.05$.

[8 Marks]

QUESTION B6 [*20 Marks*]

B6 (a) In Bayesian inference define what is meant by a conjugate prior distribution.

[3 Marks]

(b) Let $Y_1, Y_2, \ldots, Y_n$ denote a random sample from a Bernoulli distribution where

$$P(Y_i = 1) = p \text{ and } P(Y_i = 0) = 1 - p,$$

and assume that the prior distribution for p is $beta(\alpha, \beta)$, i.e.

$$f(y) = \begin{cases} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} y^{\alpha-1}(1-y)^{\beta-1}, & 0 < y < 1, \\ 0, & \text{otherwise.} \end{cases}$$

(i) Find the posterior distribution for p.

[12 Marks]

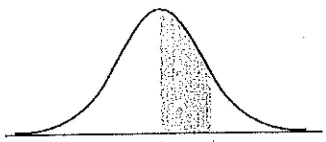
(ii) Find the Bayes estimators for p.

[5 Marks]

END OF EXAMINATION PAPER

Table AIV.2 Standard Norms Table

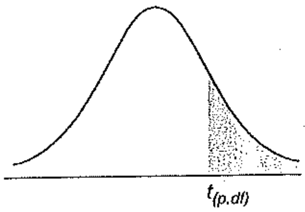
Area between 0 and *z*



z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.0000	0.0040	0.0080	0.0120	0.0160	0.0199	0.0239	0.0279	0.0319	0.0359
0.1	0.0398	0.0438	0.0478	0.0517	0.0557	0.0596	0.0636	0.0675	0.0714	0.0753
0.2	0.0793	0.0832	0.0871	0.0910	0.0948	0.0987	0.1026	0.1064	0.1103	0.1141
0.3	0.1179	0.1217	0.1255	0.1293	0.1331	0.1368	0.1406	0.1443	0.1480	0.1517
0.4	0.1554	0.1591	0.1628	0.1664	0.1700	0.1736	0.1772	0.1808	0.1844	0.1879
0.5	0.1915	0.1950	0.1985	0.2019	0.2054	0.2088	0.2123	0.2157	0.2190	0.2224
0.6	0.2257	0.2291	0.2324	0.2357	0.2389	0.2422	0.2454	0.2486	0.2517	0.2549
0.7	0.2580	0.2611	0.2642	0.2673	0.2704	0.2734	0.2764	0.2794	0.2823	0.2852
0.8	0.2881	0.2910	0.2939	0.2967	0.2995	0.3023	0.3051	0.3078	0.3106	0.3133
0.9	0.3159	0.3186	0.3212	0.3238	0.3264	0.3289	0.3315	0.3340	0.3365	0.3389
1.0	0.3413	0.3438	0.3461	0.3485	0.3508	0.3531	0.3554	0.3577	0.3599	0.3621
1.1	0.3643	0.3665	0.3686	0.3708	0.3729	0.3749	0.3770	0.3790	0.3810	0.3830
1.2	0.3849	0.3869	0.3888	0.3907	0.3925	0.3944	0.3962	0.3980	0.3997	0.4015
1.3	0.4032	0.4049	0.4066	0.4082	0.4099	0.4115	0.4131	0.4147	0.4162	0.4177
1.4	0.4192	0.4207	0.4222	0.4236	0.4251	0.4265	0.4279	0.4292	0.4306	0.4319
1.5	0.4332	0.4345	0.4357	0.4370	0.4382	0.4394	0.4406	0.4418	0.4429	0.4441
1.6	0.4452	0.4463	0.4474	0.4484	0.4495	0.4505	0.4515	0.4525	0.4535	0.4545
1.7	0.4554	0.4564	0.4573	0.4582	0.4591	0.4599	0.4608	0.4616	0.4625	0.4633
1.8	0.4641	0.4649	0.4656	0.4664	0.4671	0.4678	0.4686	0.4693	0.4699	0.4706
1.9	0.4713	0.4719	0.4726	0.4732	0.4738	0.4744	0.4750	0.4756	0.4761	0.4767
2.0	0.4772	0.4778	0.4783	0.4788	0.4793	0.4798	0.4803	0.4808	0.4812	0.4817
2.1	0.4821	0.4826	0.4830	0.4834	0.4838	0.4842	0.4846	0.4850	0.4854	0.4857
2.2	0.4861	0.4864	0.4868	0.4871	0.4875	0.4878	0.4881	0.4884	0.4887	0.4890
2.3	0.4893	0.4896	0.4898	0.4901	0.4904	0.4906	0.4909	0.4911	0.4913	0.4916
2.4	0.4918	0.4920	0.4922	0.4925	0.4927	0.4929	0.4931	0.4932	0.4934	0.4936
2.5	0.4938	0.4940	0.4941	0.4943	0.4945	0.4946	0.4948	0.4949	0.4951	0.4952
2.6	0.4953	0.4955	0.4956	0.4957	0.4959	0.4960	0.4961	0.4962	0.4963	0.4964
2.7	0.4965	0.4966	0.4967	0.4968	0.4969	0.4970	0.4971	0.4972	0.4973	0.4974
2.8	0.4974	0.4975	0.4976	0.4977	0.4977	0.4978	0.4979	0.4979	0.4980	0.4981
2.9	0.4981	0.4982	0.4982	0.4983	0.4984	0.4984	0.4985	0.4985	0.4986	0.4986
3.0	0.4987	0.4987	0.4987	0.4988	0.4988	0.4989	0.4989	0.4989	0.4990	0.4990
3.1	0.4990	0.4991	0.4991	0.4991	0.4992	0.4992	0.4992	0.4992	0.4993	0.4993
3.2	0.4993	0.4993	0.4994	0.4994	0.4994	0.4994	0.4994	0.4995	0.4995	0.4995
3.3	0.4995	0.4995	0.4995	0.4996	0.4996	0.4996	0.4996	0.4996	0.4996	0.4997
3.4	0.4997	0.4997	0.4997	0.4997	0.4997	0.4997	0.4997	0.4997	0.4997	0.4998

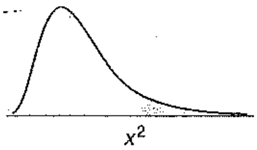
Table AIV.3 *t*-Table

Right Tail Probabilities



df\p	0.40	0.25	0.10	0.05	0.025	0.01	0.005	0.0005
1	0.324920	1.000000	3.077684	6.313752	12.70620	31.82052	63.65674	636.6192
2	0.288675	0.816497	1.885618	2.919986	4.30265	6.96456	9.92484	31.5991
3	0.276671	0.764892	1.637744	2.353363	3.18245	4.54070	5.84091	12.9240
4	0.270722	0.740697	1.533206	2.131847	2.77645	3.74695	4.60409	8.6103
5	0.267181	0.726687	1.475884	2.015048	2.57058	3.36493	4.03214	6.8688
6	0.264835	0.717558	1.439756	1.943180	2.44691	3.14267	3.70743	5.9588
7	0.263167	0.711142	1.414924	1.894579	2.36462	2.99795	3.49948	5.4079
8	0.261921	0.706387	1.396815	1.859548	2.30600	2.89646	3.35539	5.0413
9	0.260955	0.702722	1.383029	1.833113	2.26216	2.82144	3.24984	4.7809
10	0.260185	0.699812	1.372184	1.812461	2.22814	2.76377	3.16927	4.5869
11	0.259556	0.697445	1.363430	1.795885	2.20099	2.71808	3.10581	4.4370
12	0.259033	0.695483	1.356217	1.782288	2.17881	2.68100	3.05454	4.3178
13	0.258591	0.693829	1.350171	1.770933	2.16037	2.65031	3.01228	4.2208
14	0.258213	0.692417	1.345030	1.761310	2.14479	2.62449	2.97684	4.1405
15	0.257885	0.691197	1.340606	1.753050	2.13145	2.60248	2.94671	4.0728
16	0.257599	0.690132	1.336757	1.745884	2.11991	2.58349	2.92078	4.0150
17	0.257347	0.689195	1.333379	1.739607	2.10982	2.56693	2.89823	3.9651
18	0.257123	0.688364	1.330391	1.734064	2.10092	2.55238	2.87844	3.9216
19	0.256923	0.687621	1.327728	1.729133	2.09302	2.53948	2.86093	3.8834
20	0.256743	0.686954	1.325341	1.724718	2.08596	2.52798	2.84534	3.8495
21	0.256580	0.686352	1.323188	1.720743	2.07961	2.51765	2.83136	3.8193
22	0.256432	0.685805	1.321237	1.717144	2.07387	2.50832	2.81876	3.7921
23	0.256297	0.685306	1.319460	1.713872	2.06866	2.49987	2.80734	3.7676
24	0.256173	0.684850	1.317836	1.710882	2.06390	2.49216	2.79694	3.7454
25	0.256060	0.684430	1.316345	1.708141	2.05954	2.48511	2.78744	3.7251
26	0.255955	0.684043	1.314972	1.705618	2.05553	2.47863	2.77871	3.7066
27	0.255858	0.683685	1.313703	1.703288	2.05183	2.47266	2.77068	3.6896
28	0.255768	0.683353	1.312527	1.701131	2.04841	2.46714	2.76326	3.6739
29	0.255684	0.683044	1.311434	1.699127	2.04523	2.46202	2.75639	3.6594
30	0.255605	0.682756	1.310415	1.697261	2.04227	2.45726	2.75000	3.6460
∞	0.253347	0.674490	1.281552	1.644854	1.95996	2.32635	2.57583	3.2905

Table AIV.4 Chi-Square Probabilities



df\p	0.995	0.99	0.975	0.95	0.90	0.10	0.05	0.025	0.01	0.005
1	4×10^{-5}	16×10^{-5}	0.001	0.004	0.016	2.706	3.841	5.024	6.635	7.879
2	0.010	0.020	0.051	0.103	0.211	4.605	5.991	7.378	9.210	10.597
3	0.072	0.115	0.216	0.352	0.584	6.251	7.815	9.348	11.345	12.838
4	0.207	0.297	0.484	0.711	1.064	7.779	9.488	11.143	13.277	14.860
5	0.412	0.554	0.831	1.145	1.610	9.236	11.070	12.833	15.086	16.750
6	0.676	0.872	1.237	1.635	2.204	10.645	12.592	14.449	16.812	18.548
7	0.989	1.239	1.690	2.167	2.833	12.017	14.067	16.013	18.475	20.278
8	1.344	1.646	2.180	2.733	3.490	13.362	15.507	17.535	20.090	21.955
9	1.735	2.088	2.700	3.325	4.168	14.684	16.919	19.023	21.666	23.589
10	2.156	2.558	3.247	3.940	4.865	15.987	18.307	20.483	23.209	25.188
11	2.603	3.053	3.816	4.575	5.578	17.275	19.675	21.920	24.725	26.757
12	3.074	3.571	4.404	5.226	6.304	18.549	21.026	23.337	26.217	28.300
13	3.565	4.107	5.009	5.892	7.042	19.812	22.362	24.736	27.688	29.819
14	4.075	4.660	5.629	6.571	7.790	21.064	23.685	26.119	29.141	31.319
15	4.601	5.229	6.262	7.261	8.547	22.307	24.996	27.488	30.578	32.801
16	5.142	5.812	6.908	7.962	9.312	23.542	26.296	28.845	32.000	34.267
17	5.697	6.408	7.564	8.672	10.085	24.769	27.587	30.191	33.409	35.718
18	6.265	7.015	8.231	9.390	10.865	25.989	28.869	31.526	34.805	37.156
19	6.844	7.633	8.907	10.117	11.651	27.204	30.144	32.852	36.191	38.582
20	7.434	8.260	9.591	10.851	12.443	28.412	31.410	34.170	37.566	39.997
21	8.034	8.897	10.283	11.591	13.240	29.615	32.671	35.479	38.932	41.401
22	8.643	9.542	10.982	12.338	14.041	30.813	33.924	36.781	40.289	42.796
23	9.260	10.196	11.689	13.091	14.848	32.007	35.172	38.076	41.638	44.181
24	9.886	10.856	12.401	13.848	15.659	33.196	36.415	39.364	42.980	45.559
25	10.520	11.524	13.120	14.611	16.473	34.382	37.652	40.646	44.314	46.928
26	11.160	12.198	13.844	15.379	17.292	35.563	38.885	41.923	45.642	48.290
27	11.808	12.879	14.573	16.151	18.114	36.741	40.113	43.195	46.963	49.645
28	12.461	13.565	15.308	16.928	18.939	37.916	41.337	44.461	48.278	50.993
29	13.121	14.256	16.047	17.708	19.768	39.087	42.557	45.722	49.588	52.336
30	13.787	14.953	16.791	18.493	20.599	40.256	43.773	46.979	50.892	53.672
40	20.707	22.164	24.433	26.509	29.051	51.805	55.758	59.342	63.691	66.766
50	27.991	29.707	32.357	34.764	37.689	63.167	67.505	71.420	76.154	79.490
60	35.534	37.485	40.482	43.188	46.459	74.397	79.082	83.298	88.379	91.952
70	43.275	45.442	48.758	51.739	55.329	85.527	90.531	95.023	100.425	104.215
80	51.172	53.540	57.153	60.391	64.278	96.578	101.879	106.629	112.329	116.321
90	59.196	61.754	65.647	69.126	73.291	107.565	113.145	118.136	124.116	128.299
100	67.328	70.065	74.222	77.929	82.358	118.498	124.342	129.561	135.807	140.169