UNIVERSITY OF SWAZILAND

SUPPLEMENTARY EXAMINATION, 2018/2019

BASS IV, B.Ed (Sec.) IV, B.Sc IV

Title of Paper : COMPUTATIONAL METHODS

Course Number : MAT415

Time Allowed : Three (3) Hours

Instructions

1. This paper consists of SIX (6) questions in TWO sections.
2. Section A is **COMPULSORY** and is worth 40%. Answer ALL questions in this section.
3. Section B consists of FIVE questions, each worth 20%. Answer ANY THREE (3) questions in this section.
4. Show all your working.
5. Start each new major question (A1, B2 – B6) on a new page and clearly indicate the question number at the top of the page.
6. You can answer questions in any order.
7. Indicate your program next to your student ID.

Special Requirements: NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GIVEN BY THE INVIGILATOR.

SECTION A [*40 Marks*]: ANSWER ALL QUESTIONS

QUESTION A1 [*40 Marks*]

A1 (a) Consider the system $\dot{x} = x(3 - 2x - 2y)$, $\dot{y} = y(2 - 2x - y)$.

i. Write the system in **LaTeX** form without the preamble. [3 marks]

ii. Write a code in **Maple** or **Mathematica** for determining the equilibria and Jacobian of the system (i.e the commands you would use to determine the equilibria and Jacobian). [3 marks]

iii. Classify all the fixed points of the system. [4 marks]

(b) Sociologists recognize a phenomenon called social diffusion, which is the spreading of a piece of information, technological innovation, or cultural fad among a population. The members of the population can be divided into two classes: those who have the information and those who do not. In a fixed population whose size is known, it is reasonable to assume that the rate of diffusion is proportional to the number of those who have the information times the number yet to receive it. If X denotes the number of individuals who have the information in a population of N people, then a mathematical model for social diffusion is given by

$$\frac{dX}{dt} = kX(N - X)$$

where t is time in days and k is a positive constant.

i. Construct a phase line identifying the signs of X' and X''. [4 marks]

ii. Sketch representative solution curves. [3 marks]

iii. How many people eventually receive the information? [1 marks]

(c) For $r \in \mathbb{R}$, condisder the differential equation

$$\dot{x} = rx - 2x^2 + x^3$$

i. Show that $x^* = 0$ is a fixed point for any value of the parameter r, and determine its stability. Hence identify a bifurcation point r_1. [6 marks]

ii. Show that for certain values of the parameter r there are additional fixed points. For which values of r do these fixed points exist? Determine their stability and identify a further bifurcation point r_2. [6 marks]

iii. Sketch the bifurcation diagram for all values of r against x^*. [3 marks]

(d) Consider the sytem of differential equations $\dot{x} = -y + x^2$, $\dot{y} = -x + y^2$. [3 marks]

i. Does the system exhibit unique solutions? Justify your answer. [3 marks]

ii. State the location of any fixed point of the system. [1 marks]

SECTION B: ANSWER ANY *THREE* QUESTIONS

QUESTION B2 [*20 Marks*]

B2 Large solitary animals such as pandas can, at low population densities, have trouble finding mates and so may, for small N, have population growth proportional to N^2. A model for such a population is

$$\frac{dN}{dt} = rN^2\left(1 - \frac{N}{K}\right)$$

where r and K are positive parameters.

(a) Explain briefly what each of the parameters mean and what their units can be. [6 marks]

(b) Show that by suitable choice of dimensionless variables one can reduce the system the form

$$\frac{dx}{d\tau} = x^2(1 - x).$$

Your answer should include expressions for x and τ in terms of N and t and the parameters r and K. [9 marks]

(c) If the initial population is nonzero, $x(0) = x_0 > 0$, find $\lim_{\tau\to\infty} x(\tau)$ and determine whether the limiting equilibrium is stable. [5 marks]

QUESTION B3 [*20 Marks*]

B3 (a) State any two components of each of the following

i. An abstract of a thesis [2 marks]

ii. Introduction of a thesis [2 marks]

iii. Conclusion of a thesis [2 marks]

iv. A LaTeX file [2 marks]

(b) Consider the system of differential equations

$$\frac{dx}{dt} = \alpha x\left(1 - \frac{x}{K}\right) - \beta xy$$
$$\frac{dy}{dt} = \beta xy - \gamma y$$

i. Without stating the preamble, write the Jacobian matrix of the system in **LaTeX** form. [4 marks]

ii. Given that $\alpha = 0.2$, $\beta = 0.1$, $\gamma = 0.08$, $K = 1000$, $x(0) = 500$ and $y(0) = 100$, write a **MATLAB** or **OCTAVE** script for numerically solving the above system of equations using Euler's method on the time domain; $t \in [0, 10]$. [8 marks]

QUESTION B4 [*20 Marks*]

B4 (a) Consider a model of blood cholesterol levels based on the fact that cholesterol is manufactured by the body for use in the construction of cell walls and is observed from foods containing cholesterol. Let $C(t)$ be the amount (in milligrams per decilitre) of the cholesterol in the blood of a particular person at a time t in days). Then

$$\frac{dC}{dt} = k_1(N - C) + k_2 E$$

N is the persons natural cholesterol level, k_1 is a production parameter, E is the daily rate at which cholesterol is eaten and k_2 is the absorption parameter. Suppose that $N = 200$, $k_1 = 0.1$, $k2 = 0.1$, $E = 400$, and $C(0) = 150$. What will the persons cholesterol level be after 2 days on this diet? [8 marks]

(b) The atmospheric pressure, P, in kPa exponentially decreases with increasing height above sea levcel, h. The pressure can be modelled by the function

$$P(h) = 101 \times \left(\frac{25}{22}\right)^{-h}$$

where h is the height above sea level in kilometres.

i. What is the exact atmospheric pressure at sea level? [2 marks]

ii. Mount Szko has a height of 2228 metres above sea level at the top. Calculate the atmospheric pressure at the top of Mount Szko. [2 marks]

iii. Calculate the height when the atmospheric pressure is 10 kPa. [3 marks]

(c) A ball is fired from the top of a tower. The height, h, in metres of the ball above the ground is modelled by the function

$$h(t) = -2t^2 + 20t + 8, \ t >\geq 0$$

where t is the time is seconds from the moment the ball is fired.

i. Calculate the maximum height reached by the ball. [3 marks]

ii. Determine the total time the ball was above the height of the tower. [2 marks]

QUESTION B5 [*20 Marks*]

B5 The following system of differential equations models an interaction between two populations:

$$\dot{x} = x(1 - x) - \alpha x$$
$$\dot{y} = \beta xy - \gamma y$$

where α, β and γ are strictly positive constants.

(a) Briefly describe the type of interaction that this system might describe. [2 marks]

(b) Find the equilibrium points of these equations and determine the values of the parameters for which they satisfy $x \geq 0, y \geq 0$ and the values for which they are asymptotically stable. [12 marks]

(c) Sketch phase portraits of the system for the cases (i) $\gamma > \beta$ and (ii) $\gamma < \beta$. [6 marks]

QUESTION B6 [*20 Marks*]

B6 The number of aphids in a garden can be controlled by releasing ladybirds. The dynamics of aphids (A) and ladybirds (L) are given by the following ordinary differential equations:

$$\frac{dA}{dt} = A(a - bA - cL)$$
$$\frac{dL}{dt} = L(ecA - d)$$

where a, b, c, d and e are positive constants.

(a) Give ecological meanings of the parameters a, b, c, and d. [4 marks]

(b) Determine the time invariant solutions of the aphids and ladybirds. [3 marks]

(c) Which of the solutions in (b) above are stable? [5 marks]

(d) Given that $a = 2$, $b = 1$, $c = 1$, $d = 1$, $e = 1$, sketch a plausible phase portrait of the model. [8 marks]

END OF EXAMINATION