University of ESwatini

Final Examination, December 2018

B.A.S.S. , B.Sc, B.Ed

Title of Paper : Abstract Algebra II

Course Number : M423

Time Allowed : Three (3) Hours

Instructions

1. This paper consists of TWO sections.
   a. **SECTION A(COMPULSORY): 40 MARKS**
      Answer ALL QUESTIONS.
   b. **SECTION B: 60 MARKS**
      Answer ANY THREE questions.
      Submit solutions to ONLY THREE questions in Section B.
2. Begin each major question (A1, A2, B2, etc) on a new page.
3. Show all your working.
4. Special requirements: None.
5. Indicate your program next to your student ID number.

THIS PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GIVEN BY THE INVIGILATOR.

SECTION A: Answer All Questions

QUESTION A1

(a) Show that if D is an integral domain, then the ring $D[x]$ of polynomials is also an integral domain. [7 Marks]

(b) The polynomial $x^4 + 2x^3 + x^2 + x + 1$ has a linear factor in $\mathbb{Z}_3[x]$. Find the factorisation into irreducible polynomials in $\mathbb{Z}_3[x]$. [7 Marks]

(c) Show that $\alpha = \sqrt{1 - \sqrt{2}}$ is algebraic over $\mathbb{Q}$. Find the minimum polynomial and the degree of α

i. over $\mathbb{R}$

ii. over $\mathbb{Q}$

[6 Marks]

QUESTION A2

(a) Show that the polynomial $x^2 + x + 1$ is irreducible in $\mathbb{Z}_2[x]$. [6 Marks]

(b) Set α be a zero of $x^2 + x + 1$ in the extension field of $\mathbb{Z}_2$, i.e $E = \mathbb{Z}_2(\alpha)$.

i. Write down all elements of $\mathbb{Z}_2(\alpha)$. [6 Marks]

ii. Construct the multiplication table for $\mathbb{Z}_2(\alpha)$, showing the inverses for each non zero element. [8 Marks]

SECTION B: Answer Three(3) Questions Only

QUESTION B3

(a) i. Give an example of a ring R with unity 1 that has a sub-ring R' with unity $1'$ but $1 \neq 1'$. [4 Marks]

ii. Show that the set of matrices of the form $\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix}$ is a left ideal of $M_2(F)$ where $a, b \in F$. [2 Marks]

iii. Give an example of a commutative ring R with no zero divisors but is not an integral domain. [2 Marks]

(b) State whether true or false.

i. Every ring has unity.

ii. If the ring R is commutative then the ring $R[x]$ is also commutative.

iii. Every ring with unity has at most two units.

iv. If F is a field, then the units in $F[x]$ are the units in F.

v. Non zero elements of a field form a group under multiplication.

vi. If R is a ring then the divisors of zero in $R[x]$ are the divisors of zero in R.

[12 Marks]

QUESTION B4

(a) Use Fermat's theorem to compute the remainder when 8^{123} is divided by 13. [6 Marks]

(b) For each of the given algebraic numbers $\alpha \in \mathbb{C}$, find $\text{irred}(\alpha, \mathbb{Q})$ and $\deg(\alpha, \mathbb{Q})$.

i. $\sqrt{3} + i$.

ii. $\sqrt{\frac{1}{5} + \sqrt{7}}$.

[6 Marks]

(c) Show that if a polynomial $f(x) \in \mathbb{Z}[x]$ is reducible over $\mathbb{Q}$ then its also reducible over $\mathbb{Z}$. [8 Marks]

QUESTION B5

(a) Define;

i. An ideal N in a ring R.

ii. A divisor of zero in a ring R.

iii. The characteristic of a ring R.

iv. Unity in a ring R.

v. A unit in a ring R.

[10 Marks]

(b) Prove that in the ring $\mathbb{Z}_n$;

i. the divisors of zero are those elements that are not relatively prime to n.

ii. the elements that are relatively prime to n cannot be divisors of zero.

[10 Marks]

QUESTION B6

(a) Give a definition of the following:

i. an integral domain.

ii. a field.

[6 Marks]

(b) Prove that a finite domain is a field. [10 Marks]

(c) Give an example of an integral domain that is not a field. [4 Marks]

QUESTION B7

(a) State Eisenstein's criterion for irreducibility. [3 Marks]

(b) Use Eisenstein's criterion to show that $f(x) = 26x^5 - 5x^4 + 25x^2 - 10$ is irreducible over $\mathbb{Q}$. [3 Marks]

(c) Find all zeros of $x^3 + 2x + 2$ in $\mathbb{Z}_7$. [6 Marks]

(d) Find the quotient and the remainder when $f(x) = 3x^4 + 2x^2 - 1$ is divided by $D(x) = 2x^2 + 4x$ in $\mathbb{Z}_5[x]$. [8 Marks]

END OF EXAMINATION