FINAL EXAMINATION, 2017/2018
B.Sc. III, BASS III, B.Ed. III

Title of Paper : Abstract Algebra I

Course Number : MS323/MAT324

Time Allowed : Three (3) Hours

Instructions

1. This paper consists of TWO (2) Sections:

   a. SECTION A (40 MARKS)
      - Answer ALL questions in Section A.

   b. SECTION B
      - There are FIVE (5) questions in Section B.
      - Each question in Section B is worth 20 Marks.
      - Answer ANY THREE (3) questions in Section B.
      - If you answer more than three (3) questions in Section B, only the first three questions answered in Section B will be marked.

2. Show all your working.

Special Requirements: NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GIVEN BY THE INVIGILATOR.

SECTION A

ANSWER ALL QUESTIONS

QUESTION A1

(a) Prove that every subgroup of a cyclic group is cyclic. (10 marks)

(b) Give an example of a group satisfying the give conditions or, if there is no such example, say so. (Do not prove anything)

(i) An infinite cyclic group
(ii) a noncyclic group of order 4
(iii) A nonabelian cyclic group

(6 marks)

(c) Find the greatest common divisor d of the numbers 204 and 54, i.e, $d = (204, 54)$ and express d in the form $d = 204m + 54n$ for some $m, n \in Z$ (4 marks)

QUESTION A2

(a) Prove that, if $a \equiv b(\text{mod } m)$ and $c \equiv d(\text{mod } m)$, then $ac \equiv bd(\text{mod } m)$

(5 marks)

(b) (i) Give the definition of a cyclic group. (2 marks)
(ii) Prove that every finite group of prime order is cyclic. (5 marks)

(c) Determine whether the set Q, with respect to the binary operation $a * b = a + b - 2018$ is a group. (8 marks)

QUESTION B3

Let $\alpha = \begin{pmatrix} 1\,2\,3\,4\,5\,6\,7\,8 \\ 5\,8\,6\,3\,2\,4\,1\,7 \end{pmatrix}$ and $\beta = \begin{pmatrix} 1\,2\,3\,4\,5\,6\,7\,8 \\ 1\,7\,5\,3\,8\,2\,6\,4 \end{pmatrix}$

(a) Express α and β as products of disjoint cycles, and then as products of transpositions. For each of them, say whether it is an even permutation or an odd one. (8 marks)

(b) Compute a^{-1}, $\beta^{-1}\alpha$, $(\alpha\beta)^{-1}$ (6 marks)

(c) Solve the equations $\alpha x = \beta$, $y\alpha = \beta$ (3 marks)

(d) Find the order of β and compute β^{2018} ((3 marks)

QUESTION B4

(a) Let $\varphi : G \rightarrow H$ be isomorphism of groups

(i) Prove that if e_g and e_H are the identity elements of G and H respectively, then $(e_g)\varphi = e_H$ (4 marks)

(ii) Prove that for any $a \in G$

$$(a^{-1})\varphi = [(a)\varphi]^{-1}$$ (4 marks)

(b) Let $H\{\rho_0, \rho_1, \rho_2\}$ and $G = S_3$ where $\rho_o = (1)$ – identity $\rho_1 = (\ 1\,2\,3)$ $\rho_2 = (\ 1\,3\,2)$

Show that H is a normal subgroup of G. (12 marks)

QUESTION B5

(a) Define the term subgroup of a group G. (3 marks)

(b) State (do not prove), Lagrange's theorem for finite groups. (3 marks)

(ii) What order subgroups can possibly exist? (Justify your answers) (8 marks)

(d) Does an element of order 3 exist in S_3? If so, use it to given an example of a subgroup of order 3 in S_3 (6 marks)

QUESTION B6

(a) Prove that if G is a group of order P, where P is prime, then G is cyclic. (8 marks)

(b) Prove that every cyclic group is abelian. (6 marks)

(c) Let m be a positive integer greater than 1, and let, for $a, b \in Z$

aRb If and only if $a \equiv b(\text{mod } m)$

Prove that R is an equivalence relation on Z (6 marks)

QUESTION B7

(a) Let $H = \langle 6 \rangle$ be the subgroup of Z_{18} generated by 6.

(i) Find all cosets of H in Z_{18}

(ii) Write the group table for the factor/quotient group Z_{18}/H (10 marks)

(b) In the following pairs the two groups are not isomorphic. In each case given a reason why

(i) Z_5 , Z_6

(ii) Z_6 , S_3

(4 marks)

(c) Solve the following:

(i) $9x \equiv 11(\text{mod } 36)$

(ii) $3x + 1 \equiv 3(\text{mod } 5)$ (6 marks)