UNIVERSITY OF SWAZILAND

SUPPLEMENTARY EXAMINATION, 2017/2018

B.Sc. IV, BASS IV, BED. IV

Title of Paper : Abstract Algebra II

Course Number : M423

Time Allowed : Three (3) Hours

Instructions

1. This paper consists of TWO (2) Sections:

   a. SECTION A (40 MARKS)
      - Answer **ALL** questions in Section A.

   b. SECTION B
      - There are FIVE (5) questions in Section B.
      - Each question in Section B is worth 20 Marks.
      - Answer **ANY THREE (3)** questions in Section B.
      - If you answer more than three (3) questions in Section B, **only the first three questions answered in Section B will be marked.**

2. Show all your working.

Special Requirements: NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GIVEN BY THE INVIGILATOR.

SECTION A: ANSWER ALL QUESTIONS

QUESTION A1

(a) Find the greatest common divisor of the polynomials

$$f(x) = x^4 + 4x^3 + 7x^2 + 6x + 2 \quad \text{and}$$
$$g(x) = x^3 + 4x^2 + 7x + 4$$

over $\mathbb{Q}$ and express it as a linear combination of $f(x)$ and $g(x)$.

(8 marks)

(b) Prove that if R is a ring with unity and N is an ideal of R containing a unit, then $N = R$.

(6 marks)

(c) Describe all units in each of the following rings:

(i) $\mathbb{Z}_7$

(ii) $\mathbb{Z} \times \mathbb{Q} \times \mathbb{Z}_3$

(6 marks)

QUESTION A2

(a) State whether or not each of the given function ν, is an Euclidean valuation for the given integral domain

(i) $\nu(n) = n^2$ for non-zero $n \in \mathbb{Z}$,

(ii) $\nu(a) = So$ for non-zero values $a \in \mathbb{Q}$.

(8 marks)

(b) State Kroneckers theorem. (Do not prove) (4 marks)

(c) Given that every element β of $E = F(\propto)$ can be uniquely expressed in the form

$$\beta = b_0 + b_1 \propto + b_2 \propto^2 + \ldots + b_{n-1} \propto^{n-1}$$

where each of $b_i \in F$, $\propto$ algebraic over the field F and $\deg(\propto, F) \geq 1$. Show that if F is finite with q elements, then $E = F(\propto)$ has q^n elements.

(8 marks)

SECTION B: ANSWER ANY THREE QUESTIONS

QUESTION B3

(a) Use Fermat's theorem to compute the remainder when 8^{123} is divided by 13.
(6 marks)

(b) For each of the given algebraic numbers $\alpha \in C$ find irred$(\propto, \mathbb{Q})$ and $\deg(\propto, \mathbb{Q})$.

(i) $\sqrt{3} + i$

(i) $\sqrt{\frac{1}{5} + \sqrt{7}}$

(6 marks)

(c) Show that if a polynomial $f(x) \in \mathbb{Z}[x]$ is reducible over $\mathbb{Q}$ then its also reducible over $\mathbb{Z}$

QUESTION B4

(a) Show that for a field F, the set of all matrices of the form

$$\begin{pmatrix} a_{11} & a_{12} \\ 0 & 0 \end{pmatrix} \qquad a_{ij} \in F$$

is a right ideal but not a left ideal of $M_2(F)$ (6 marks)

(b) Let $\varphi_{\alpha}\ \mathbb{Z}_7[x] \to \mathbb{Z}_7$. Evaluate each of the following for the indicated evaluation homomorphism:

(i) $\varphi_2\left(3x^{79} + 5x^{53} + 2x^{43}\right)$

(ii) $\varphi_3\left[\left(x^3 + 2\right)\left(4x^2 + 3\right)\left(x^7 + 3x^2 + 1\right)\right]$

(10 marks)

(c) Show that if D is an integral domain, then $D[x]$ is also an integral domain.

(4 marks)

QUESTION B5

(a) Prove that every field is an integral domain. (6 marks)

(b) Which of the following are rings with the usual addition and multiplication

(i) $\{a + b\sqrt{z} : a, b \in \mathbb{Z}\}$

(ii) $M_2(\mathbb{R})$ with zero determinant. (8 marks)

(c) Mark each of the following true or false:

(i) Every finite integral domain is a field
(ii) Every division ring is commutative
(iii) $\mathbb{Z}_6$ is not an integral domain

(6 marks)

QUESTION B6

(a) Classify each of the given $\propto \in \mathbb{C}$ as algebraic or transcedental over the given field F. If $\propto$ is algebraic over F, find $\deg(\propto, F)$.

(i) $\propto = 1 + i$, $F = \mathbb{Q}$

(ii) $\propto = \sqrt{\pi}$, $F = \mathbb{Q}[\pi]$

(iii) $\propto = \pi^2$, $F = \mathbb{Q}$

(iv) $\propto = \pi^2$, $F = \mathbb{Q}(\pi^3)$

(v) $\propto = \pi^2$, $F = \mathbb{Q}(\pi)$

(10 marks)

(b) Show that the ring $\mathbb{Z}_2 \times \mathbb{Z}_2$ is not a field. (5 marks)

(c) Find a polynomial of $\deg ree > 0$ in $Z_4[x]$ that is a unit. (5 marks)

QUESTION B7

(a) Suppose F is a field, f is an irreducible polynomial over F and g, h are polynomials over F such that f divides gh. Show that either f divides g or f divides h. (10 marks)

(b) Define an ideal N of a ring R (2 marks)

(c) Find all ideals of $\mathbb{Z}_{10}$ and all maximal ideals of $\mathbb{Z}_{18}$. (8 marks)