UNIVERSITY OF SWAZILAND

SUPPLEMENTARY EXAMINATION, 2016/2017

BASS III, B.Ed (Sec.) III, B.Sc. III

Title of Paper : Real Analysis

Course Number : M331

Time Allowed : Three (3) Hours

Instructions

1. This paper consists of SIX (6) questions in TWO sections.
2. Section A is **COMPULSORY** and is worth 40%. Answer ALL questions in this section.
3. Section B consists of FIVE questions, each worth 20%. Answer ANY THREE (3) questions in this section.
4. Show all your working.
5. Start each new major question (A1, B2 – B6) on a new page and clearly indicate the question number at the top of the page.
6. You can answer questions in any order.

Special Requirements: NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GIVEN BY THE INVIGILATOR.

SECTION A [*40 Marks*]: ANSWER ALL QUESTIONS

QUESTION A1 [*40 Marks*]

(a) Consider a set $E \subseteq \mathbb{R}$. Give precise definitions for each of the following.

i. $m \in \mathbb{R}$ is a *lower bound* of E. (2)

ii. $M \in \mathbb{R}$ is an *upper bound* of E. (2)

iii. E is *bounded*. (2)

iv. m is the *minimum* of E. (2)

v. M is the *maximum* of E. (2)

vi. M is the *supremum* of $E \neq \emptyset$. (2)

vii. m is the *infimum* of $E \neq \emptyset$. (2)

(b) Let

$$A = \left\{ \frac{1}{n} \,.\, n \in \mathbb{N} \right\}.$$

i. Find (if they exist) $\min(A)$, $\max(A)$, $\inf(A)$, and $\sup(A)$. (4)

ii. Is A bounded? Explain. (2)

(c) Give the $\varepsilon - \delta$ definition for $\lim_{x \to a} f(x) = L$, where f is a real-valued function. (3)

(d) Give the $\varepsilon - N$ definition for $\lim_{n \to \infty} x_n = x$ where $\{x_n\}$ is a sequence of real numbers. (3)

(e) i. Define a Cauchy sequence. (3)

ii. State the Cauchy convergence criterion for sequences. (3)

iii. Define a Cauchy series. (2)

(f) Prove or disprove.

i. *If f is continuous at c, then f is differentiable at c.* (3)

ii. *If $\lim_{n \to \infty} x_n = 0$, then the series $\sum_{n=1}^{\infty} x_n$ is convergent.* (3)

END OF SECTION A – TURN OVER

SECTION B: ANSWER ANY *THREE* QUESTIONS

QUESTION B2 [20 *Marks*]

(a) Determine whether or not the series $\sum_{n=1}^{\infty} \frac{2n}{3n+1}$ is convergent. (4)

(b) Use the integral test to show that the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent. (6)

(c) Let $\{x_n\}$ be the sequence recursively defined by

$$x_1 = 2, \quad x_{n+1} = x_n - \frac{x_n^2 - 2}{2x_n}.$$

i. Show that $x_n > 0$ for all integers $n \geq 1$. (4)

ii. Show that $\{x_n\}$ is a non-increasing sequence. (4)

iii. Determine whether the sequence converges or not. Give reasons for your answer. (2)

QUESTION B3 [20 *Marks*]

(a) Give a precise $\varepsilon - N$ argument to show that

$$\lim_{n\to\infty} \frac{2n^2}{5n^2+1} = \tfrac{2}{5}.$$

(7)

(b) Show that the sequence $\{x_n\}$ given by

$$x_n = \frac{1}{n}$$

is Cauchy. (6)

(c) i. Fill in the blanks:
A sequence $\{x_n\}$ diverges to ∞ if for every ______ there is ______ such that ______ whenever ______. (4)

ii. Let $x_n = n^3$. Use (i) above to show that $x_n \to \infty$ as $n \to \infty$. (3)

QUESTION B4 [20 *Marks*]

(a) Show that the equation

$$\ln x = 2 - x$$

has a solution in the interval $[1, e]$. (6)

(b) Let $f(x) = 10x - 11$. Use an $\varepsilon - \delta$ argument to show that $\lim_{x \to 5} f(x) = 39$. (7)

(c) Prove: *If* $\lim_{x \to c} f(x)$ *exists, then it is unique.* (7)

QUESTION B5 [20 *Marks*]

(a) Prove: *If* $f : I \to \mathbb{R}$ *is differentiable at* $c \in I$, *then* f *is continuous at* c. (7)

(b) State the Fundamental Theorem of Calculus. (3)

(c) Let $f : [0, 2] \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} 1, & \text{if } x < 1, \\ \frac{1}{2}, & \text{if } x = 1, \\ 0, & \text{if } x > 1. \end{cases}$$

Show that f is Riemann integrable and find $\int_0^2 f(x)\, dx$. (10)

QUESTION B6 [20 *Marks*]

(a) Prove: *Let* $\{x_n\}$ *and* $\{y_n\}$ *be convergent sequences. Then*

$$\lim_{n \to \infty} (x_n + y_n) = \lim_{n \to \infty} x_n + \lim_{n \to \infty} y_n.$$

(7)

(b) Let $f(x) = x^2$ and let $0 < a < \infty$. Show that f is uniformly continuous on $[-a, a]$. (7)

(c) Prove the reverse triangle inequality

$$\left| |x| - |y| \right| \leq |x - y|, \quad \text{for } x, y \in \mathbb{R}.$$

(6)

END OF EXAMINATION PAPER