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University of Swaziland

Final Examination, 2011/2012

BSc IV, Bass IV, BEd IV

Title of Paper : Abstract Algebra II

Course Number : M423

Time Allowed : Three (3) hours

Instructions :

1. This paper consists of SEVEN questions.
2. Each question is worth 20%.
3. Answer ANY FIVE questions.
4. Show all your working.

THIS PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GIVEN BY THE INVIGILATOR.

QUESTION 1

1.1 Which of the following sets is a ring with the usual operations of addition and multiplication? In each case, either prove that it is a ring or explain why it is not.

1.1.1 The set of all pure imaginary numbers ri for $r \in \mathbb{R}$.

1.1.2 The set $\mathbb{Z}[\sqrt{5}] = \{a + b\sqrt{5} : a, b \in \mathbb{Z}\}$.

[10

1.2 Let R be a ring with unity and let I be an ideal in R containing a unit. Prove that $I = R$. [7]

1.3 Use the result of 1.2 to show that a field contains no proper ideals. [3]

QUESTION 2

2.1 Explain what is meant by saying that a polynomial $f(x) \in \mathbb{Z}[x]$ is *irreducible over* $\mathbb{Z}$. [2]

2.2 State Eisenstein's criterion for irreducibility and use it to show that

$$26x^5 - 5x^4 + 25x^2 - 10$$

is irreducible over $\mathbb{Q}$. [2,

2.3 Find $q(x)$ and $r(x)$ as described by the division algorithm so that $f(x) = q(x)g(x) + r(x)$ with $r(x) = 0$ or $\deg(r(x)) < \deg(g(x))$ for

$$f(x) = x^5 - 2x^4 + 3x - 5, \quad g(x) = 2x + 1 \quad \text{in } Z_{11}[x].$$

[6]

2.4 Find all zeroes of $x^3 + 2x + 2$ in $\mathbb{Z}_7$. [6]

QUESTION 3

3.1 Define

3.1.1 a divisor of zero in a ring R, [2]

3.1.2 a unit in a ring R with unity, [2]

3.1.3 the characteristic of a ring R, [2]

3.1.4 a ring homomorphism, [2]

3.1.5 an ideal in a ring R. [2]

3.2 Prove that in the ring $\mathbb{Z}_n$,

3.2.1 the divisors of zero are those nonzero elements that are not relatively prime to n, [5]

3.2.2 the elements that are relatively prime to n cannot be divisors of zero. [5]

QUESTION 4

4.1 Let R be a ring. Prove that a subset S of R is a subring of R if and only if (1) S is closed with respect to the ring operations on R, and (2) for each element a in S, $-a$ is also in S. [10]

4.2 Let R be a commutative ring. For an arbitrary element p in R, form the set

$$P = \{pr : r \in R\}.$$

Prove that P is an ideal in R. [10]

QUESTION 5

5.1 Let α be a zero of $x^2 + x + 1$ in an extension field of $\mathbb{Z}_2$.

5.1.1 Write down all the elements of $\mathbb{Z}_2(\alpha)$. [4]

5.1.2 Construct the multiplication table for $\mathbb{Z}_2(\alpha)$. [Show how each product was found.] [8]

5.2 For each algebraic number $\alpha \in \mathbb{C}$, find $\text{irr}(\alpha, \mathbb{Q})$ and $\deg(\alpha, \mathbb{Q})$.

5.2.1 $\sqrt{2} + i$ [4]

5.2.2 $\sqrt{2 + \sqrt{3}}$ [4]

QUESTION 6

6.1 Let α be an element of an extension field E of F. Explain what it means for α to be

6.1.1 *algebraic over* F? [2]

6.1.2 *transcendental over* F? [1]

6.2 State (do not prove) Kronecker's Theorem. [4]

6.3 Consider the polynomial $x^3 + x^2 + 1$ in $\mathbb{Z}_2[x]$.

6.3.1 Show that $x^3 + x^2 + 1$ is irreducible over $\mathbb{Z}_2$. [3]

6.3.2 Let α be a zero of $x^3 + x^2 + 1$ in an extension field of $\mathbb{Z}_2$. Show that $x^3 + x^2 + 1$ factors into three linear factors in $\mathbb{Z}_2(\alpha)[x]$ by finding this factorisation. [10]

QUESTION 7

7.1 Define

7.1.1 an integral domain, [2]

7.1.2 a field. [2]

7.2 Give an example of an integral domain that is not a field. [2]

7.3 Prove that every finite integral domain is a field. [8]

7.4 Determine whether or not $\mathbb{Q}[x]/\langle x^3 - 6x + 6\rangle$ is a field. Justify your answer. [6]

END OF EXAMINATION PAPER