UNIVERSITY OF SWAZILAND

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FINAL EXAMINATION 2011/12

BSC. III

TITLE OF PAPER	:	COMPLEX ANALYSIS
COURSE NUMBER	:	M313
TIME ALLOWED	:	THREE (3) HOURS
INSTRUCTIONS	:	1. THIS PAPER CONSISTS OF SEVEN QUESTIONS. 2. ANSWER ANY FIVE QUESTIONS
SPECIAL REQUIREMENTS	:	NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GRANTED BY THE INVIGILATOR.

QUESTION 1

(a) (i) What is a principal value of arg z?

(ii) Solve $z^3 = i - 1$. [2,4]

(b) In the complex plane define and give an example of

(i) $\epsilon-$ neighborhood of point z_o,

(ii) simply connected set of points. [2,2]

(c) Sketch the following sets and determine which are domains

(i) $| z - 2 + i| < 1$,

(ii) $Imz > 1$,

(iii) $0 < arg\ z \leq \frac{\pi}{4}$. [2,2,2]

(d) Construct a line

$$Im\frac{1}{z+i} = 1.$$

[4]

QUESTION 2

(a) Find a region into which a transformation $w = z^2$ maps a square

$0 \leq Re\ z \leq 1, \quad 0 \leq Imz \leq 1.$ [4]

(b) Find the limits. Give your reasonings

(i) $\lim_{z \to \infty} \frac{2z+i}{z+1}$,

(ii) $\lim_{z \to 3} \frac{1}{(z-3)^2}$. [2,2]

(c) Define uniformely continous function of complex variable. [2]

(d) Using just a definition of derivative, find if possible $f'(z)$ if

(i) $f(z) = 3z^2$,

(ii) $f(z) = Re\ z$.

[2,2]

(e) Derive Cauchy-Riemann conditions. [6]

QUESTION 3

(a) Using Cauchy-Riemann equations (CRE)

(i) State the sufficient conditions theorem for existance of $f'(z)$, and thus

(ii) check if there is $f'(z)$ if $f(z) = z^3$, $\quad f(z) = \frac{1}{z}$. Find $f'(z)$. [2,6]

(b) Verify CRE for

$f(z) = x^2 + 2iy$, where $z = x + iy$. [3]

(c) (i) Derive CRE in polars,

(ii) Consider $f(z) = \frac{1}{z}$. Pass to polar coordinates to check if there is $f'(z)$ and find it. [4,5]

QUESTION 4

a) Consider $f = |z|^2$. Give your reasonings to answer if f

(i) is analytic,

(ii) has the singular points. [2,2]

b) Prove that $f(z) = u(x,y) + iv(x,y)$, where $z = x + iy$, is analytic in domain D, if and only if $v(x,y)$ is harmonic conjugate function (HCF) of $u(x,y)$. [6]

(c) For the function $u(x,y) = 4xy(y^2+x^2)$, find whether it can be the real part of a complex analytic function, and if so, find the corresponding imaginary part. [3]

(d) Show that $u(x,y) = x^2 - y^2 - 2xy - 2x + 3y$ is harmonic and find corresponding HCF [7]

QUESTION 5

a) Let $f(z) = \frac{z+2}{z}$. Evaluate $\int_c f(z)dz$, if c is positively oriented semicircle $z = 2e^{i\theta}, \quad 0 \le \theta \le \pi$. [6]

b) (i) Derive Cauchy formula for continuous $f'(z)$.

HINT: Apply Green's formula

$$\int_c P dx + Q dy = \int\int_R (Q_x - P_y) dx dy.$$

(ii) Use result from (i) to evaluate

$$\int_c \frac{dz}{(z^2+1)(z^2+16)},$$

if $c = \{z : |z| = 3 \text{ in positive direction and } |z| = 2 \text{ in negative direction}\}$ [6,3]

(c) Apply Cauchy integral formula to evaluate

$\int_c \frac{zdz}{(25-z^2)(z+i)}$, where c is a positively oriented circle $|z| = 2$. [5]

QUESTION 6

a) Let C be a simple closed contour, described in the positive sense in the z plane, and write

$g(a) = \int_c \frac{z^3+2z}{(z-a)^3} dz$. Show

(i) $g(a) = 6\pi i a$, when a is inside c, and

(ii) $g(a) = 0$ when a is outside c. [4,2]

b) (i) State the Taylor series theorem and thus

(ii) expand $\frac{1}{1-z}$ in a Maclaurin series for $|z| < 1$. [2,3]

(c) Expand $\frac{1}{(1-z)(2+z)}$ in Laurent series in powers of z valid for

(i) $|z| < 1$,

(ii) $1 < |z| < 2$,

(iii) $2 < |z| < \infty$. [3,3,3]

HINT: $\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n, \quad |z| < 1$.

QUESTION 7

a) For $f(z) = \frac{1}{z + z^2}$

(i) find residue at $z = 0$, and thus

(ii) evaluate $\int_c \frac{dz}{z + z^2}$, where c is a positively oriented circle $|z| = \frac{1}{2}$. [4,3]

b) (i) State the residue theorem, and

(ii) apply it to evaluate

$\int_c \frac{z+1}{z^2 - 2z} dz$, where c is a positively oriented circle $|z| = 3$. [2,4]

c) Using the residue theorem evaluate

$\int_{-\infty}^{\infty} \frac{\cos 3x}{(x^2 + 1)^2} dx.$ [7]