134

UNIVERSITY OF SWAZILAND

FINAL EXAMINATION 2010

BSc. IV/ B.A.S.S. IV/ BEd IV

TITLE OF PAPER : PARTIAL DIFFERENTIAL EQUATIONS

COURSE NUMBER : M415

TIME ALLOWED : THREE (3) HOURS

INSTRUCTIONS :

1. THIS PAPER CONSISTS OF SEVEN QUESTIONS.
2. EACH QUESTION IS WORTH 20%.
3. ANSWER ANY FIVE QUESTIONS.
4. SHOW ALL YOUR WORKING.

THIS PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GIVEN BY THE INVIGILATOR.

135

Question 1

(a) Find the values of a, b and c if

$$u(x,y) = e^{ax} f(bx + cy + u^3)$$

is a solution to the partial differential equation

$$u_x - (1 + 3u^3)u_y - u = 0.$$

[10 marks]

(b) Find a particular solution of the given partial differential equation that passes through the given curve Γ

$$xu_x - 2x^2 u_y = u$$
$$\Gamma \;:\; u = x^2 + x \quad \text{on} \quad y = 2x + 1.$$

[10 marks]

Question 2

(a) Determine the region in which the given partial differential equations are elliptic, hyperbolic and parabolic

i. $\sin^2 x u_{xx} - 2y \sin x + y^2 u_{yy} = u.$ **[3 marks]**

ii. $e^x u_{xx} + e^y u_{yy} + e^{x+y} u_x = 0.$ **[3 marks]**

(b) Find the characteristics for the following partial differential equation

$$xu_{xx} + u_{yy} = x^2.$$

[6 marks]

(c) Reduce the following partial differential equation to canonical form and find the general solution

$$u_{xx} + 5u_{xy} + 6u_{yy} = 0.$$

[8 marks]

Question 3

Consider the equation

$$u_{tt} - c^2 u_{xx} = 0$$
$$u(x,0) = f(x)$$
$$u_t(x,0) = g(x)$$

(a) Find the characteristics. [3 marks]

(b) Reduce the partial differential equation to its canonical form. [8 marks]

(c) Hence or otherwise show that the solution is given by

$$u(x,t) = \frac{f(x-ct) + f(x+ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\tau)d\tau.$$

[9 marks]

Question 4

Consider the function

$$f(x) = \begin{cases} \frac{1}{2}(-\pi - x), & -\pi \leq x < 0; \\ 0, & x = 0; \\ \frac{1}{2}(\pi - x), & 0 < x \leq \pi. \end{cases}$$

(a) Find the fourier series expansion for $f(x)$. [10 marks]

(b) Use Parseval's identity to find the value of the sum

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

[10 marks]

Question 5

(a) Show that the initial value problem with non-homogeneous boundary conditions

$$\begin{aligned} &u_t - u_{xx} = 0, && 0 < x < L \\ &u(0,t) = T_1, && t > 0 \\ &u(L,t) = T_2, && t > 0 \\ &u(x,0) = f(x),, && 0 \leq x \leq L \end{aligned}$$

can be transformed to an initial value problem with homogeneous boundary conditions. [6 marks]

(b) Find the solution of the transformed problem using separation of variables. [8 marks]

(c) Hence or otherwise find the solution for the initial value problem

$$\begin{aligned} &u_t - u_{xx} = 0, && 0 < x < 8 \\ &u(0,t) = 4, && t > 0 \\ &u(8,t) = 12, && t > 0 \\ &u(x,0) = x,, && 0 \leq x \leq 8 \end{aligned}$$

[6 marks]

Question 6

Solve the following equations using the method of Laplace transforms

(a)

$$\begin{aligned} &u_t = u_x + u, \qquad x > 0, \ \ t > 0 \\ &u(x,0) = e^{-4x}, \qquad x \geq 0 \\ &\lim_{x\to\infty} |u(x,t)| < \infty, \qquad t > 0 \end{aligned}$$

[10 marks]

(b)

$$\begin{aligned} &u_t + x u_x = x, \qquad x > 0, \ \ t > 0 \\ &u(x,0) = 0, \qquad x \geq 0 \\ &u(0,t) = 0, \qquad t \geq 0 \end{aligned}$$

[10 marks]

Question 7

Solve the Dirichlet problem inside the circle

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0, \quad 0 < r < 1, \quad -\pi < \theta < \pi$$

$$u(1,\theta) = \begin{cases} 1, & -\pi < x < 0; \\ 2, & 0 < x < \pi. \end{cases}$$

[20 marks]

Table of Laplace Transforms

$f(t)$	$F(s)$
t^n	$\frac{n!}{s^{n+1}}$
$\frac{1}{\sqrt{t}}$	$\sqrt{\frac{\pi}{s}}$
e^{at}	$\frac{1}{s-a}$
$t^n e^{at}$	$\frac{n}{(s-a)^{n+1}}$
$\frac{1}{a-b}\left(e^{at}-e^{bt}\right)$	$\frac{1}{(s-a)(s-b)}$
$\frac{1}{a-b}\left(ae^{at}-be^{bt}\right)$	$\frac{s}{(s-a)(s-b)}$
$\sin(at)$	$\frac{a}{s^2+a^2}$
$\cos(at)$	$\frac{s}{s^2+a^2}$
$\sin(at)-at\cos(at)$	$\frac{2a^3}{(s^2+a^2)^2}$
$e^{at}\sin(at)$	$\frac{b}{(s-a)^2+b^2}$
$e^{at}\cos(at)$	$\frac{s-a}{(s-a)^2+b^2}$
$\sinh(at)$	$\frac{a}{s^2-a^2}$
$\cosh(at)$	$\frac{s}{s^2-a^2}$
$\sin(at)\sinh(at)$	$\frac{2a^2}{s^4+4a^4}$
$\frac{d^n f}{dt^n}(t)$	$s^n F(s) - s^{n-1}f(0) - \cdots - f^{(n-1)}(0)$