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UNIVERSITY OF SWAZILAND

FINAL EXAMINATION 2010/2011

BSc. /BEd. /B.A.S.S III

TITLE OF PAPER	:	REAL ANALYSIS
COURSE NUMBER	:	M 331
TIME ALLOWED	:	THREE (3) HOURS
INSTRUCTIONS	:	1. THIS PAPER CONSISTS OF SEVEN QUESTIONS.
		2. ANSWER ANY FIVE QUESTIONS
SPECIAL REQUIREMENTS	:	NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GRANTED BY THE INVIGILATOR.

QUESTION 1

1. (a) Let S be a set of real numbers. Explain precisely what is meant by each of the following statements.

    i. A real number α is an upper bound for S. [2 marks]

    ii. A real number β is a lower bound for S. [2 marks]

    iii. S is bounded above. [2 marks]

    iv. S is bounded below. [2 marks]

    v. S is bounded. [2 marks]

   (b) Determine whether the set $S := \{x \in \mathbb{R} : |x| + |x+1| < 2\}$ is bounded or not. [4 marks]

   (c) Prove that the sum of a rational number and an irrational number is always irrational. [3 marks]

   (d) Let S be a non-empty set of real numbers, and let $\alpha = \sup S$. Also, let $T := \{y \in \mathbb{R} : y = -x \text{ and } x \in S\}$. Show that $\inf T = -\alpha$. [3 marks]

QUESTION 2

2. (a) Let (x_n) be a sequence of real numbers. Explain precisely what is meant by each of the following statements.

    i. A real number l is a limit of the sequence (x_n). [3 marks]

    ii. The sequence (x_n) is convergent. [2 marks]

    iii. The sequence (x_n) is bounded. [2 marks]

   (b) Show that if a sequence (x_n) of real numbers is convergent then (x_n) is bounded. [4 marks]

   (c) i. Explain precisely what it means to say that "*a sequence* (x_n) *of real numbers is Cauchy*". [3 marks]

    ii. State the Cauchy convergence criterion for a sequence of real numbers. [2 marks]

    iii. Show that if (x_n) and (y_n) are Cauchy sequences, then $(x_n + y_n)$ is a Cauchy sequence. [4 marks]

QUESTION 3

3. (a) Let $f, g : [a, b] \rightarrow \mathbb{R}$ be functions, and let $c \in (a, b)$.

   i. Explain precisely what it means to say that f is continuous at c. [2 marks]

   ii. If f is continuous at c, then show that the function $|f| : [a, b] \rightarrow \mathbb{R}$ defined by $|f|(x) := |f(x)|$ is also continuous at c. [4 marks]

   iii. Prove that if both f and g are continuous at c then the sum $f + g$ is also continuous at c. [4 marks]

   iv. Is the converse of part 3(a)iii above true? Justify your answer. [2 marks]

   (b) i. State the Intermediate value theorem. [2 marks]

   ii. Show that the equation $x = \cos x$ has a solution in the interval $[0, \pi/2]$. [3 marks]

   (c) Is the following statement true or false? Justify your answer. If a function $f : (-1, 0) \rightarrow \mathbb{R}$ is continuous then f is bounded. [3 marks]

QUESTION 4

4. (a) Let $f : (a, b) \rightarrow \mathbb{R}$ be a function.

   i. Explain what is meant by saying that f is differentiable at $c \in (a, b)$. [2 marks]

   ii. Show that f is continuous at $c \in (a, b)$ whenever f is differentiable at point c. [4 marks]

   iii. Give an example of a function $f : (0, 2) \rightarrow \mathbb{R}$ that is continuous at some point $c \in (0, 2)$ and yet f not differentiable at c. [2 marks]

   (b) i. State the Mean value theorem. [2 marks]

   ii. Use the Mean value theorem to prove each of the following statements.

   A. $|\sin x| \leq |x|,\ \forall x \in \mathbb{R}$. [5 marks]

   B. $\dfrac{x-1}{x} < \log x < x - 1$ for $x > 1$. [5 marks]

QUESTION 5

5. (a) Let $\sum a_n$ be a series in $\mathbb{R}$. Then, explain the following statements.
   i. The k-th partial sum. [2 marks]
   ii. $\sum a_n$ converges. [2 marks]
   iii. $\sum a_n$ is absolutely convergent. [1 marks]

   (b) State the Cauchy convergence criterion for series. [2 marks]

   (c) Prove that $\sum a_n$ converges whenever $\sum a_n$ is absolutely convergent. [4 marks]

   (d) Let $\sum a_n$ be a convergent series of non-negative numbers, and let (b_n) be a bounded sequence of real numbers. Then, show that the series $\sum a_n b_n$ converges. [5 marks]

   (e) Determine whether the series

$$\sum \frac{(2n)!}{3^n (n!)^2}$$

   converges or diverges. State any theorems used. [4 marks]

QUESTION 6

6. (a) State Riemann's criterion for integrability. [2 marks]

   (b) Prove that if $f : [a, b] \to \mathbb{R}$ is bounded and $\{P_n : n \in \mathbb{N}\}$ is a sequence of partitions of $[a, b]$ such that $\lim_n (U(P_n; f) - L(P_n; f)) = 0$ then f is integrable. [4 marks]

   (c) Use part 6b above to show that the function $f : [0, 1] \to \mathbb{R}$ defined by

$$f(x) := \begin{cases} 0, & \text{if } 0 \leq x < 1/2 \\ 1, & \text{if } 1/2 \leq x < 1 \end{cases}$$

   is Riemann integrable and $\int_0^1 f = \frac{1}{2}$. [8 marks]

   (d) Determine whether each of the following statements is true or false. Justify your answers.
   i. Every function $f : [0, 1] \to \mathbb{R}$ is Riemann integrable whenever f is bounded. [3 marks]
   ii. There are two distinct functions $f, g : [0, 1] \to \mathbb{R}$ such that the product fg is Riemann integrable and yet neither f nor g is Riemann integrable. [3 marks]

QUESTION 7

7. (a) Let $f, g, h : (a, b) \to \mathbb{R}$ be functions and let $c \in (a, b)$. Show that if

   i. $f(x) \leq g(x) \leq h(x)$,

   ii. $f(c) = g(c) = h(c)$, and

   iii. both f and h are continuous at c,

   then g is also continuous at c. [6 marks]

   (b) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is differentiable and that $|f'(x)| < 1, \forall x \in \mathbb{R}$. Then, prove that the equation

$$f(x) = x$$

   has at most one solution. [6 marks]

   (c) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is twice differentiable on $\mathbb{R}$ and that $a, b \in \mathbb{R}$ with $a < b$. Let $g, h : \mathbb{R} \to \mathbb{R}$ be functions defined by

$$g(x) := f(b) - f(x) - (b - x)f'(x)$$

   and

$$h(x) := (b - a)^2 g(x) - (b - x)^2 g(a)$$

   i. Show that $h(a) = h(b)$. [2 marks]

   ii. Use Rolle's theorem to show that for some $c \in (a, b)$

$$f(b) = f(a) + (b - a)f'(a) + \frac{1}{2}(b - a)^2 f''(c)$$

   [6 marks]