University of Swaziland

Final Examination, May 2011

BSc III, Bass III, BEd III

Title of Paper : Abstract Algebra I

Course Number : M323

Time Allowed : Three (3) hours

Instructions :

1. This paper consists of SEVEN questions.
2. Each question is worth 20%.
3. Answer ANY FIVE questions.
4. Show all your working.

THIS PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GIVEN BY THE INVIGILATOR.

QUESTION 1

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(a) Define a group. [4]

(b) Determine whether the set

$$G = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a, b \in \mathbb{R}, a \neq 0 \right\}$$

gives a group structure under matrix multiplication [8]

(c) Prove that a subset H of a group G is a subgroup of G if and only if $H \neq \emptyset$, and whenever $g, h \in H$, then $gh^{-1} \in H$. [8]

QUESTION 2

(a) Find all $x \in \mathbb{Z}$ such that

$$3x \equiv 2(\text{mod } 7).$$

[4]

(b) Define an equivalence relation on a set S. [4]

(c) Define a relation $\sim$ on $\mathbb{Z}$ by $m \sim n$ if and only if $m \equiv n(\text{mod } 4)$.

i. Show that $\sim$ is an equivalence relation on $\mathbb{Z}$. [8]

ii. Describe the partition given by $\sim$. [4]

QUESTION 3

(a) Let H be the subset

$$\{\rho_0 = (1), \rho_1 = (123), \rho_2 = (132)\}$$

of the symmetric group S_3.

i. Show that H is a subgroup of S_3. [5]

ii. Show that H is cyclic. [5]

(b) Prove that every cyclic group is abelian. [5]

(c) Show that $\mathbb{Z}_p$ has no proper subgroups if p is a prime number. [5]

QUESTION 4

(a) Find all the subgroups of $\mathbb{Z}_{18}$ and give a lattice diagram. [10]

(b) Let $\phi : G \to H$ be a group isomorphism and let e be the identity of G. Prove that $\phi(e)$ is the identity in H and that $[\phi(a)]^{-1} = \phi(a^{-1})$. [10]

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QUESTION 5

Let $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 6 & 8 & 1 & 7 & 5 & 3 & 4 & 2 \end{pmatrix}$ and $\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 7 & 1 & 8 & 3 & 4 & 5 & 2 & 6 \end{pmatrix}$.

(a) i. Express α and β as products of disjoint cycles. [4]

ii. Express α and β as products of transpositions and indicate whether they are even or odd permutations. [4]

(b) Compute

i. α^{-1} [2]

ii. $\beta^{-1}\alpha$ [3]

iii. $(\alpha\beta)^{-1}$ [3]

(c) Prove that every group of prime order is cyclic [4]

QUESTION 6

(a) For $a, b, m \in \mathbb{Z}$, show that if $\gcd(a, m) = 1$ and $\gcd(b, m) = 1$, then $\gcd(ab, m) = 1$. [7]

(b) Find integers r and s such that $\gcd(211, 130) = 211r + 130s$. [7]

(c) Find the number of genrators in each of the cyclic groups $\mathbb{Z}_{30}$ and $\mathbb{Z}_{42}$. [6]

QUESTION 7

(a) Define a normal subgroup of a group. [4]

(b) Verify that the subgroup $N = \{(1), (123), (132)\}$ is a normal subgroup of the group S_3. [6]

(c) For each binary operation $*$ defined on the given set, say whether or not $*$ gives a group structure with the set.

i. Define $*$ on $\mathbb{Q}^+$ by $a * b = ab/2$, for all $a, b \in \mathbb{Q}^+$. [4]

ii. Define $*$ on $\mathbb{R}$ by $a * b = ab + a + b$ for all $a, b \in \mathbb{R}$. [6]

END OF EXAMINATION PAPER