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UNIVERSITY OF SWAZILAND

FINAL EXAMINATIONS 2010/2011

BSc. / BEd. / B.A.S.S. II

TITLE OF PAPER	:	DYNAMICS I
COURSE NUMBER	:	M 255
TIME ALLOWED	:	THREE (3) HOURS
INSTRUCTIONS	:	1. THIS PAPER CONSISTS OF SEVEN QUESTIONS. 2. ANSWER ANY FIVE QUESTIONS
SPECIAL REQUIREMENTS	:	NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GRANTED BY THE INVIGILATOR.

QUESTION 1

(a) For which values of b are the vectors

$\mathbf{v} = (-6, b, 2)$ and $\mathbf{w} = (b, b^2, b)$ orthogonal? [3]

(b) Find the vector in the same direction as

$\mathbf{v} = (-2, 4, 2)$ but which has a length of 6. [3]

(c) Find the projection of the vector

$\mathbf{A} = \mathbf{i} - 2\mathbf{j} + \mathbf{k}$ onto $\mathbf{B} = 4\mathbf{i} - 4\mathbf{j} + 7\mathbf{k}$. [5]

(d) (i) Prove that $\nabla\phi$ is a vector perpendicular to the surface $\phi(x, y, z) = c$, where c is a constant. [5]

(ii) Find a unit vector that is normal to the surface $2x^2 + 4yz - 5z^2 = -10$ at the point $(3, -1, 2)$. [4]

QUESTION 2

A parametrization of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1; \qquad a, b > 0,$$

traversed in the counterclockwise direction is given by $x = a\cos t$, $y = b\sin t$; $t \geq 0$. Suppose that a particle moves along this ellipse in the counterclockwise direction. Find:

(a) The position vector $\mathbf{r}$; [1]

(b) the velocity vector $\mathbf{v}$; [1]

(c) the speed $|\mathbf{v}|$; [1]

(d) the acceleration vector $\mathbf{a}$; [1]

(e) the magnitude of the acceleration $|\mathbf{a}|$; [1]

(f) the unit tangent vector $\hat{\mathbf{T}}$; [2]

(g) the principal unit normal vector $\hat{\mathbf{N}}$; [5]

(h) the curvature κ; [2]

(i) the unit binormal vector $\hat{\mathbf{B}}$; and [2]

(j) the tangential and normal components of the acceleration of the particle at the point $P\left(\frac{-a}{\sqrt{2}}, \frac{b}{\sqrt{2}}\right)$. [4]

QUESTION 3

(a) In spherical coordinates (ρ, ϕ, θ), the position vector of an arbitrary point (x, y, z) is given by

$$\mathbf{r} = \rho \sin\phi \cos\theta \hat{\mathbf{i}} + \rho \sin\phi \sin\theta \hat{\mathbf{j}} + \rho \cos\phi \hat{\mathbf{k}}.$$

Find:

(i) $\hat{\rho}$; [2]

(ii) $\hat{\phi}$; [2]

(iii) $\hat{\theta}$; and [2]

(iv) the velocity vector $\mathbf{v}$ [3]

for any particle moving in this coordinate system.

(b) Let $a(t) = a_1(t)\hat{\mathbf{i}} + a_2(t)\hat{\mathbf{j}} + a_3(t)\hat{\mathbf{k}}$ be a differentiable vector function, and let $\phi(t)$ be a differentiable scalar function. Prove that

$$\frac{\mathrm{d}(\phi \mathbf{a})}{\mathrm{d}t} = \phi \frac{\mathrm{d}\mathbf{a}}{\mathrm{d}t} + \mathbf{a}\frac{\mathrm{d}\phi}{\mathrm{d}t}.$$

[4]

(c) If $\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$ and $r = |\mathbf{r}|$, show that:

(i) $\nabla r = \dfrac{\mathbf{r}}{r}$,

(ii) $\nabla^2(\log r) = \dfrac{1}{r^2}$. [3,4]

QUESTION 4

(a) From a point O, at height h above sea level, a particle is projected under gravity with a velocity of magnitude $\frac{3}{2}\sqrt{gh}$. Find the two possible angles of projection if the particle strikes the sea at horizontal distance $3h$ from 0. [10 marks]

(b) Two points A and B are at distance d apart. A particle starts from A and moves in the direction $\overrightarrow{AB}$ with initial velocity u and uniform acceleration a. A second particle starts at the same time from B and moves in the direction $\overrightarrow{BA}$ with initial velocity $2u$ and retardation a.

(i) Prove that the particles collide at time $\frac{d}{3u}$ from the beginning of the motion. [5 marks]

(ii) Prove that if the particles collide before the second particle returns to B, then

$$ad < 12u^2.$$

[5 marks]

QUESTION 5

(a) A particle is projected with velocity **u** from a point O in a vertical plane through the line of greatest slope of a plane inclined at an angle $-\beta$ to the horizontal. After time T, the particle strikes the inclined plane at the point P, at a distance R from O. If **u** makes an angle α with the horizontal, and if $|\mathbf{u}| = u$, show that:

(i) $T = \dfrac{2u\sin(\alpha+\beta)}{g\cos\beta}$ and $R = \dfrac{u^2[sin(2\alpha+\beta)+\sin\beta]}{g\cos^2\beta}$;

(ii) for constant u and β, R is maximum when $\alpha = \dfrac{\pi}{4} - \dfrac{\beta}{2}$. [9,3]

(b) A particle of unit mass moves subject to a central force. Determine the law of force if the path followed by the particle is $r = a\cos\theta$, where a is a constant. [8]

QUESTION 6

(a) Let $x(t) = c_1\cos(\omega t + \phi_1)$ and $y(t) = c_2\cos(\omega t + \phi_2)$ be harmonic functions in standard form with the same angular frequency ω. What do we mean by x leads y, and when does x lag y? [1]

(b) State whether x leads or lags y in each of the following:

(i) $x = 2\cos(2t + \frac{\pi}{4})$, $y = 3\cos(2t + \frac{9\pi}{2})$ [2]

(ii) $x = \cos(3t)$, $y = \sin(3t)$. [2]

(c) Express $A\cos(\omega t) + B\sin(\omega t)$ in the standard form $C\cos(\omega t + \phi)$ when $A = 3\frac{1}{2}$ and $B = -1$. [2]

(d) Find the current $I(t)$ in an RLC−circuit with $R = 100$ ohms, $L = 0.1$ henries, and $C = 10^{-3}$ farads, which is connected to a source of voltage $E(t) = 155\sin 377t$, assuming zero charge and current when $t = 0$. [13]

QUESTION 7

(a) The position of a particle moving along the x axis is determined by the equation

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 4\frac{\mathrm{d}x}{\mathrm{d}t} + 8x = 20\cos(2t).$$

If the particle starts from rest at $x = 0$, find

(i) x as a function of t,

(ii) the amplitude, period, and frequency after a long time. [7,3]

(b) The weight on a vibrating spring undergoes forced vibrations according to the equation

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 4x = 8\sin(\omega t),$$

where x is the displacement from the equilibrium position and ω is a constant. If $x = 0$ and $\frac{\mathrm{d}x}{\mathrm{d}t} = 0$ when $t = 0$, find:

(i) x as a function of t,

(ii) the period of the external force for which resonance occurs. [8,2]

END OF EXAMINATION