UNIVERSITY OF SWAZILAND

SUPPLEMENTARY EXAMINATION 2008/9

BSc. /BEd. /B.A.S.S II

TITLE OF PAPER	:	LINEAR ALGEBRA
COURSE NUMBER	:	M 220
TIME ALLOWED	:	THREE (3) HOURS
INSTRUCTIONS	:	1. THIS PAPER CONSISTS OF SEVEN QUESTIONS. 2. ANSWER ANY FIVE QUESTIONS. 3. Non-programmable calculators may be used.
SPECIAL REQUIREMENTS	:	NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GRANTED BY THE INVIGILATOR.

QUESTION 1

1. (a) Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given by

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+y \\ 3y \\ 2x-y \end{pmatrix}$$

Find the matrix of T

(i) with respect to the standard bases,

(ii) with respect to B' and B where

$$B' = \left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\} \text{ and } B = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\}$$

[10 marks]

(b) Prove that

(i) $[M_i(\alpha)] = M_i\left(\frac{1}{\alpha}\right), \alpha \neq 0.$

(ii) $(H_{ij})^{-1} = H_{ij}.$

(iii) $[A_{ij}(\alpha)]^{-1} = A_{ij}(-\alpha).$ [10 marks]

QUESTION 2

2. (a) (i) Show that the vector $\begin{pmatrix} -3 \\ 12 \\ 12 \end{pmatrix}$ is a linear combination of

$$\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 4 \end{pmatrix}, \text{ and } \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix}$$

[5 marks]

(ii) Show that the set of vectors

$$\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \text{ and } \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}.$$

is a basis for $\mathbb{R}^3$. [5 marks]

(b) Let $\{v_1, v_2, \ldots, v_n\}$ be a set of non-zero vectors in a vector space V. Prove that S is linearly independent if and only if one of the vectors v_j; $j = 1, 2, \ldots, n$ is a linear combination of the preceding vectors in S. [10 marks]

QUESTION 3

3. (a) Verify the Cayley-Hamilton theorem for the matrix

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & -1 & 5 \\ 3 & 2 & 1 \end{pmatrix}$$

[5 marks]

(b) Prove that the set $B = \{x^2 + 1, x - 1, 2x + 2\}$ is a basis for the vector space $P_2(x)$, where $P_2(x)$ is the set of all polynomials of degree less than or equal to 2 and the zero polynomial. [8 marks]

(c) Define the following.

(i) A symmetric matrix.

(ii) An orthogonal matrix. [4 marks]

QUESTION 4

4. (a) Determine whether the following sets of vectors in the vector space $P_2(x)$ are linearly dependent or linearly independent. For those that are linearly dependent express the last vector as a linear combination of the rest.

(i) $\{2x^2 + 2, x^2 + 3, x\}$. [4 marks]

(ii) $\{2x^2 + x + 1, 3x^2 + x - 5, x + 13\}$. [6 marks]

(b) Given system of linear equations

$$\begin{aligned} x + y - 4z &= 0 \\ 2x + 3y + z &= 1 \\ 4x + 7y + \lambda z &= \mu, \end{aligned}$$

find conditions on λ and μ for which the system has

(i) a unique solution,

(ii) infinitely many solutions. [10 marks]

QUESTION 5

5. (a) (i) Show that each eigenvector of a square matrix A is associated with only one eigenvalue. [5 marks]

(ii) Show that $\begin{pmatrix} 0 & 0 & 5 \\ 0 & 0 & -1 \\ -5 & 1 & 0 \end{pmatrix}$ is a skew symmetric matrix. [5 marks]

(b) (i) Use Cramer's rule to solve

$$\begin{pmatrix} 2 & 2 & 3 \\ 4 & 7 & 7 \\ 4 & -2 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 7 \end{pmatrix}$$

(ii) Use Gaussian elimination to solve

$$\begin{aligned} 3x + 2y + z &= 2 \\ 4x + 2y + 2z &= 8 \\ x - y + z &= 4 \end{aligned}$$

QUESTION 6

6. (a) For

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & -1 & 5 \\ 3 & 2 & 1 \end{pmatrix}$$

find a sequence of elementary matrices $E_1, E_2, \ldots, E_n$ such that $E_1 E_2 \ldots E_n A = I$ (i.e $A^{-1} = E_n, E_{n-1}, \ldots, E_2, E_1$) and $A = E_1^{-1}, E_2^{-1}, \ldots, E_{n-1}^{-1}, E_n^{-1}$. [10 marks]

(b) Find the characteristic polynomial, eigenvectors and eigenvalues of the matrix

$$\begin{pmatrix} 2 & 2 & 3 \\ 1 & 2 & 1 \\ 2 & -2 & 1 \end{pmatrix}$$

[10 marks]

QUESTION 7

7. (a) Let $B' = \{v_1, v_2, v_3\}$ and $B = \{u_1, u_2, u_3\}$ be bases in $\mathbb{R}^3$, where

$$v_1 = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix},$$

$$u_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad u_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad u_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Find the transition matrix from B' to B. [10 marks]

(b) For which k does the following system have non-trivial solutions?

$$\begin{aligned} kx_1 + 2x_2 - x_3 &= 0 \\ (k+1)x_1 + kx_2 + 0x_3 &= 0 \\ -x_1 + kx_2 + kx_3 &= 0 \end{aligned}$$

[10 marks]