University of Swaziland

Supplementary Examination 2008

BSc III, Bass III, BEd III

Title of Paper : Complex Analysis

Course Number : M313

Time Allowed : Three (3) hours

Instructions :

1. This paper consists of SEVEN questions.
2. Each question is worth 20%.
3. Answer ANY FIVE questions.
4. Show all your working.

THIS PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GIVEN BY THE INVIGILATOR.

Question 1

(a) Find all roots of the polynomial

$$P(z) = z^4 + z^2 + 1$$

and express in the form $a + ib$. [8 marks]

(b) Consider the real function $u = e^{1+2y} \cos 2x$.

(i) Show that u is harmonic. [4 marks]

(ii) Find the harmonic conjugate of u. [4 marks]

(iii) Hence find the analytic complex function $f(z) = u + iv$ and express in terms of z. [4 marks]

Question 2

(a) Prove that

$$\tfrac{1}{2} \sin \alpha + \tfrac{1}{2^2} \sin 2\alpha + \tfrac{1}{2^3} \sin 3\alpha + \cdots = \frac{2 \sin \alpha}{5 - 4 \cos \alpha}.$$

[8 marks]

(b) Use the theory of residues to evaluate

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)(x^2+4)}.$$

[12 marks]

Question 3

(a) Consider the complex function $f(z) = \dfrac{z - i}{z + i}$.

(i) Find the first five non-zero terms of the Taylor expansion of $f(z)$ about $z = i$. [10 marks]

(ii) Determine the radius of convergence of the series obtained in (i). [2 marks]

(b) Evaluate

$$\int_{-1-i}^{1+i} \left(\frac{\bar{z}}{z^2}\right) \mathrm{d}z$$

along the circular path $x^2 + y^2 = 2$. [8 marks]

Question 4

(a) Evaluate

$$\lim_{z \to 0} \left(\frac{\sin z}{z}\right)^{1/z^2}$$ [7 marks]

(b) Solve for all values of z satisfying

$$e^{1-2iz} = 1 - i\sqrt{3},$$

and express in the form $a + ib$. [7 marks]

(c) Evaluate $\int_0^{\pi} z e^{-iz} \mathrm{d}z$ along any path. [6 marks]

Question 5

(a) Find two Laurent series expansions of $f(z) = \dfrac{1}{z(4 - z^2)}$ in powers of z, stating the region of validity in each case.

[5 marks, 5 marks]

(b) Use the theory of residues to evaluate

$$\int_0^{2\pi} \frac{\mathrm{d}\alpha}{2 + \sin \alpha}.$$ [10 marks]

Question 6

(a) Derive the formula

$$\sec^{-1} z = -i \ln \left(\frac{1 + \sqrt{1 - z^2}}{z} \right), \qquad [10 \text{ marks}]$$

where the principal branch is chosen to be the one for which $\sec 0 = 1$.

(b) Evaluate

$$\int_{\Lambda} \frac{z^2 - \cos^2 z}{(z^2 - 9)(z^2 + \pi^2)} \mathrm{d}z,$$

where Λ is the circle $|z| = 2$ traversed once positively.
null [7 marks]

(c) Find the principal value of $\omega = \left(\frac{2i}{i - \sqrt{3}} \right)^i$ and express in the form $a + ib$. [6 marks]

Question 7

(a) Your friend claims that

> *If $\int_{\Lambda} f(z)\mathrm{d}z = 0$, it implies that the function $f(z)$ is analytic everywhere on and inside the simple closed curve Λ.*

Is this statement right or wrong. Discuss. [10 marks]

(b) Evaluate

$$\oint_{\Gamma} \frac{\cos 2z}{z^{2n+1}} \mathrm{d}z, \qquad [10 \text{ marks}]$$

where Ω is the unit circle $|z| = 1$ traversed once positively, and $n \in \mathbb{Z}^+$