UNIVERSITY OF SWAZILAND

SUPPLEMENTARY EXAMINATION 2007

BSc./ B.Ed./ BASS IV

TITLE OF PAPER: ABSTRACT ALGEBRA II

COURSE NUMBER: M423

TIME ALLOWED: THREE HOURS

INSTRUCTIONS:

1. This paper consists of SEVEN questions on FOUR pages.
2. Answer any FIVE questions.
3. Calculators may be used.

SPECIAL REQUIREMENTS: NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GRANTED BY THE INVIGILATOR.

Question 1

(a) Which of the following are rings under addition and multiplication? Give reasons

(a) $\mathbb{Z}[\sqrt{-7}] = \{a + b\sqrt{-7} : a, b \in \mathbb{Z}\}$

[5 marks]

(b) $\mathbb{N} = \{1, 2, 3, \cdots\}$

[5 marks]

(c) the set of all 2×2 matrices of the form $\begin{pmatrix} a & b \\ c & 2d \end{pmatrix}$ where a, b, c, d are integers

[5 marks]

(d) $(\mathbb{Z}, +, \cdot)$ where $+$ is the usual addition and the multiplication is $a \cdot b = 0 \quad \forall a, b \in \mathbb{Z}$

[5 marks]

Question 2

(a) Which of the following are integral domains and which are fields? Justify your answer

(a) $\mathbb{Z}_2 \times \mathbb{Z}_2$

[5 marks]

(b) $\{a + bi : a, b \in \mathbb{Q}\}$

[5 marks]

(c) $\mathbb{Z} \times \mathbb{R}$

[5 marks]

(d) $\mathbb{R}[x]$

[5 marks]

Question 3

(a) Consider the polynomial $f(x) = x^3 + 2x + 3$ over $\mathbb{Z}_5[x]$
(i) Is $f(x)$ an irreducible polynomial over $\mathbb{Z}_5[x]$? Why?

[5 marks]

(ii) Express $f(x)$ as a product of irreducible polynomials of $\mathbb{Z}_5[x]$

[5 marks]

(b) Determine which of the following polynomials in $\mathbb{Z}[x]$ satisfy an Einstein criterion for irreducibility over $\mathbb{Q}$
(i) $4x^{10} - 9x^3 + 24x - 18$

[5 marks]

(ii) $2x^{10} - 25x^3 + 10x^2 - 30$

[5 marks]

Question 4

(a) For each of the given algebraic numbers $\alpha \in \mathbb{C}$ find $err(\alpha, \mathbb{Q})$ and $deg(\alpha\mathbb{Q})$
(i) $\sqrt{3 - \sqrt{6}}$

[3 marks]

(ii) $\sqrt{\frac{1}{3} + \sqrt{7}}$

[3 marks]

(iii) $\sqrt{2} + i$

[3 marks]

(b) Show that the polynomial $x^P + a$ in $\mathbb{Z}_p[x]$ is not irreducible for any $a \in \mathbb{Z}_p$ and p is prime.

[5 marks]

(c) Let α be a zero of $x^2 + x + 1$ in the extension field of $\mathbb{Z}_2$. Give the addition and multiplication tables for the elements of $\mathbb{Z}_2(\alpha)$.

[6 marks]

Question 5

(a) Give the formal definition of a Euclidean ring R.

[4 marks]

(b) Let R be a Euclidean ring.

(i) Prove that any two elements a and b in R have a greatest common divisor d.

[5 marks]

(ii) Show that there exists $m, n \in R$ such that $d = am + bn$.

[5 marks]

(c) Prove that every finite integral domain is a field.

[6 marks]

Question 6

(a) Find all ideals N and all maximal ideals p of $\mathbb{Z}_{18}$

[5 marks]

(b) In a ring $\mathbb{Z}_n$ show that

(i) divisors of zero are those elements that are NOT relatively y prime to n.

[5 marks]

(ii) elements that are relatively prime cannot be zero divisors

[5 marks]

(c) Describe all units in the ring

$$\mathbb{Z} \times \mathbb{Q} \times \mathbb{Z}$$

[5 marks]

Question 7

(a) Show that for a field F_1 the set of all matrices of the form

$$\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \quad \text{for} \quad a, b \in F$$

is a right ideal but not a left ideal of $M_2(F)$ where $M_2(F)$ - all 2×2 matrices with entries from the field F.

[6 marks]

(b) Let $\varphi_\alpha : \mathbb{Z}_7[x] \rightarrow \mathbb{Z}_7$. Evaluate each of the following for the indicated evaluation homomorphism

(i) $\varphi_5[(x^3+2)(4x^2+3)(x^7+3x^2+1)]$

[5 marks]

(ii) $\varphi_4[3x^{106}+5x^{99}+2x^{53}]$

[5 marks]

(c) Show that the rings $\mathbb{Z}$ and $3\mathbb{Z}$ are NOT isomorphic

[4 marks]

******** *END OF EXAMINATION* ********