UNIVERSITY OF SWAZILAND

FINAL EXAMINATIONS 2006/7

BSc. / BEd. / B.A.S.S. III

TITLE OF PAPER	:	NUMERICAL ANALYSIS I
COURSE NUMBER	:	M 311
TIME ALLOWED	:	THREE (3) HOURS
INSTRUCTIONS	:	1. THIS PAPER CONSISTS OF SEVEN QUESTIONS. 2. ANSWER ANY FIVE QUESTIONS
SPECIAL REQUIREMENTS	:	NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GRANTED BY THE INVIGILATOR.

QUESTION 1

1. (a) Convert the following binary numbers to their decimal equivalent

   i. $(10101.101)_2$ [4 Marks]

   ii. $(.101010101\ldots)_2$ [4 Marks]

   (b) Convert the following decimal numbers to their binary equivalent

   i. 22.640625 [4 Marks]

   ii. $\frac{3}{5}$ [4 Marks]

   (c) How can the following function be re-written to avoid problems due to loss of precision?

$$f(x) = \sqrt{x^2+4} - 2$$

[4 Marks]

QUESTION 2

2. (a) The iteration $x_{n+1} = 2 - (1+c)x_n + cx_n^3$ will converge for sufficiently close x_0 to $s = 1$ for some values of c. Find the values of c for which this is true. For what value of c will the convergence be quadratic? [8 Marks]

   (b) Show that $s = \sqrt{a}$ is a fixed point for the iterative scheme

$$x_{n+1} = \frac{x_n(x_n^2+3a)}{3x_n^2+a} \quad n \geq 0$$

   Also determine the order of this method together with its asymptotic error constant. [12 Marks]

QUESTION 3

3. Consider the function $f(x) = x^3 + 4x^2 - 10$.

   (a) Show that $f(x)$ has exactly one root in [1, 2]. [6 marks]

   (b) By performing 4 iterations of the bisection method, show that this root lies in the interval [1.3125, 1.375]. [8 marks]

   (c) How many iterations would be required to locate this zero to a tolerance of 10^{-5}? [6 marks]

QUESTION 4

4. (a) Evaluate the integral $\int_0^2 \ln(1+x)\,dx$ by trapezoidal rule with accuracy of 0.05. [12 marks]

(b) Consider a function $f(x)$ with the following values known

x	0	1	3	4
$f(x)$	2	2	2	14

Find the Lagrange polynomial through all the points. [8 marks]

QUESTION 5

5. (a) Factor the matrix

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & -1 & 6 \\ 2 & 1 & 0 \end{pmatrix}$$

into LU decomposition and use it to solve the linear system

$$\begin{aligned} x + 2y + 3z &= 4 \\ x - y + 6z &= -1 \\ 2x + y &= 0 \end{aligned}$$

[12 marks]

(b) Use the definition of the derivative at x_0 to show that if h is sufficiently small, then

$$f'(x_0) \approx \frac{1}{h}\Delta f(x_0).$$

Extend this argument to show that

$$f''(x_0) \approx \frac{1}{h^2}\Delta^2 f(x_0).$$

QUESTION 6

6. (a) Write Newton's algorithm to evaluate the square root $\sqrt{c}$ for any $c > 0$. [4 marks]

   (b) Apply the algorithm in (a) above to find $\sqrt{3}$ with the accuracy of four digits after the decimal point. Use $x_0 = 1$ as initial guess. [4 marks]

   (c) Show that the Newton's iteration for finding the reciprocal of a real number c, is

$$x_{n+1} = x_n(2 - cx_n).$$

   and show that the iteration converges for

$$x_0 \in \left[\frac{1}{2c}, \frac{3}{2c}\right]$$

[8 marks]

   (d) Show that the iteration function of the equation

$$x = \frac{a}{6 + ax} \qquad x \in [0, 1]$$

   satisfies the assumptions of the fixed point theorem for $0 \leq a < 6$. [4 marks]

QUESTION 7

7. (a) Use the two-point Gaussian Quadrature rule,

$$\int_{-1}^{1} f(x)\, dx \approx f\left(\frac{-\sqrt{3}}{3}\right) + f\left(\frac{\sqrt{3}}{3}\right),$$

   to approximate the integral

$$\int_0^1 x^2 e^{-x}\, dx$$

   and compare your result against the exact value of the integral. [10 marks]

   (b) Evaluate the integral $\int_0^1 xe^{-x}\, dx$ analytically correct to four decimal places. Use the trapezoidal rule with $h = 0.2$ and the Simpson's rule with $h = 0.25$ to compute the same integral. Compare the errors. [10 marks]