THE UNIVERSITY OF SWAZILAND

Department of Mathematics

Final Examination 2006

M423
ABSTRACT ALGEBRA II

Three (3) hours

INSTRUCTIONS
1. This paper contains SEVEN questions.
2. Answer any FIVE questions.

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GRANTED BY THE INVIGILATOR.

Question 1. (a)[*7 marks*] Find the highest common factor of 36630 and 539 and express it in the form $36630s + 539t$ where s, t are integers.

(b)[*6 marks*] Show that for any integers a, b

$$(a+b, a+2b) = (a, b).$$

(c) [*7 marks*] What is meant by saying that a polynomial $f(x) \in \mathbb{Z}[x]$ is *irreducible*? State Eisenstein's test for irreducibility, and use it to show that $26x^5 - 5x^4 + 25x^2 - 10$ is irreducible in $\mathbb{Q}[x]$.

Question 2. [*5 + 5 + 5 +5 marks*] Which of the following is a ring (with the usual operations)? In each case either prove that it is a ring or explain why it is not.

(i) the set of 2×2 matrices of the form

$$\begin{pmatrix} a & 5b \\ c & d \end{pmatrix}$$

where a, b, c, d are integers;

(ii) the set of rational numbers of the form $\frac{a}{3^b}$ where a, b are integers and $b \geq 0$;

(iii) the set $\{1, -1, i, -i\}$;

(iv) the set $\mathbb{Z}[\sqrt{5}] = \{a + b\sqrt{5} : a, b \in \mathbb{Z}\}$.

(You may assume that $\mathbb{C}$ and $M_2(\mathbb{R})$, the set of all 2×2 matrices are both rings.)

Question 3. (a) [*6 marks*] What is (i) an *integral domain*; (ii) a *field* ? Give an example of an integral domain that is not a field.

(b) [*6 marks*] Show that a finite integral domain is a field.

(c) [*8 marks*] Explain what is meant by the ring $\mathbb{Z}_n$ where $n > 1$ is an integer. Show that $\mathbb{Z}_n$ is a field if and only if n is prime.

Question 4. (a) [*7 marks*] Show that if D is an integral domain then the ring $D[x]$ of polynomials in x with coefficients in D is an integral domain but not a field.

(b) [*7 marks*] The polynomial $x^4 + 2x^3 + x^2 + x + 1$ has a linear factor in $\mathbb{Z}_3[x]$. Find its factorization into irreducible polynomials in $\mathbb{Z}_3[x]$.

(c) [*6 marks*] Show that $\alpha = \sqrt{1 - \sqrt{2}}$ is algebraic over $\mathbb{Q}$. Find the minimum polynomial and the degree of α (i) over $\mathbb{R}$; (ii) over $\mathbb{Q}$.

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Question 5. (a)[*3 marks*] Define each of the following for u, r in a ring with unity.

(i) u is a unit;

(ii) r is prime;

(iii) r is irreducible.

(b) [*3 marks*] Show that in any integral domain a prime is irreducible.

(c) [*3 marks*] Let d be a square free integer; define the *norm* $N(\alpha)$ of an element $\alpha \in \mathbb{Z}[\sqrt{d}]$ and state without proof its main properties.

(d) (i) [*4 marks*] Show that ± 1 are the only units in $\mathbb{Z}[\sqrt{-5}]$.

(ii) [*4 marks*] Show that $(2 + \sqrt{-5})$ is irreducible in $\mathbb{Z}[\sqrt{-5}]$.

(iii) [*3 marks*] By considering the product $(2+\sqrt{-5})(2-\sqrt{-5})$ show that $(2+\sqrt{-5})$ is not prime in $\mathbb{Z}[\sqrt{-5}]$.

Question 6. (a) [*7 marks*] Let R and S be rings. What is meant by (i) an *ideal* of R; (ii) a *ring homomorphism* $\theta : R \rightarrow S$?

Define the *kernel* $\ker\theta$ of a ring homomorphism $\theta : R \rightarrow S$ and show that it is an ideal of R.

(b) [*7 marks*] Which, if any, of the following is a ring homomorphism? Find the kernel for those that are homomorphisms.

(i) $\theta : \mathbb{Z} \rightarrow \mathbb{Z}$ defined by $\theta(z) = 3z$;

(ii) $\theta : R \rightarrow \mathbb{C}$ defined by $\theta\begin{pmatrix} a & b \\ -b & a \end{pmatrix} = a + ib$ where R is the ring of matrices of the form $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ (You need not prove that R is a ring.)

(c) [*6 marks*] Let $\varphi : \mathbb{Q}[x] \rightarrow \mathbb{Q}$ be the homomorphism defined by $\varphi(f) = f(1)$. Show that $\ker\varphi = \{f \in \mathbb{Q}[x] : f \text{ is divisible by } x - 1\}$.

Question 7. (a) [*3 marks*] Show that the polynomial $x^2 + x + 1$ is irreducible in $\mathbb{Z}_2[x]$.

(b) Suppose that E is an extension field of $\mathbb{Z}_2$ and $\alpha \in E$ is a root of x^2+x+1.

(i) [*2 marks*] What is meant by the field $\mathbb{Z}_2(\alpha)$?

(ii) [*2 marks*] What is meant by the *minimum polynomial* of α over $\mathbb{Z}_2$? Explain why this is $x^2 + x + 1$.

(iii) [*6 marks*] Show that every element of $\mathbb{Z}_2(\alpha)$ can be written uniquely as $a + b\alpha$ with $a, b \in \mathbb{Z}_2$.

(iv) [*7 marks*] Draw up the multiplication table for $\mathbb{Z}_2(\alpha)$ and identify the multiplicative inverse of each non-zero element.

(Any theorems you use about divisibility and HCFs in $\mathbb{Z}_2[x]$ should be stated clearly but not proved.)

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