THE UNIVERSITY OF SWAZILAND

121

Department of Mathematics

Final Examination 2005

M431
METRIC SPACES

Three (3) hours

INSTRUCTIONS
1. This paper contains SEVEN questions.
2. Answer any FIVE questions.

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GRANTED BY THE INVIGILATOR.

Question 1. (a)[*10 marks*] What is meant by saying that (X,d) is a *metric space*?

Let d be the function defined on $\mathbb{R}^2$ by

$$d(x,y) = 2|x_1 - y_1| + 3|x_2 - y_2|$$

where $x = (x_1, x_2)$ and $y = (y_1, y_2)$. Prove carefully that $(\mathbb{R}^2, d)$ is a metric space.

(b) [*10 marks*] Describe the *uniform metric* and the L_1-*metric* on the set $C[a,b]$ of continuous functions defined on the interval $[a,b]$.

Let $x(t) = t$ and $y(t) = \sin t$ for $-\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$. Calculate the distance between x and y in $C[-\frac{\pi}{2}, \frac{\pi}{2}]$ (You may assume without proof that $\sin t \leq t$ for $t \in [0, \frac{\pi}{2}]$

(i) in the uniform metric;

(ii) in the L_1-metric.

Question 2. (a) [*6 marks*] Let (X,d) be a metric space and (x_n) be a sequence in X. What is meant by saying that (x_n) is *convergent*? Show that if $x_n \to x$ as $n \to \infty$ and $a \in X$ then $d(x_n, a) \to d(x,a)$.

(b)[*4 marks*] Decide whether or not the following sequence is convergent in the usual (Euclidean) metric on $\mathbb{R}^2$.

$$x_n = \left(\cos((n^2 + \frac{1}{n})\pi), n^{\frac{1}{n}}\right)$$

(c) [*10 marks*] Explain what is meant by *pointwise convergence* of a sequence of functions (x_n) in $C[a,b]$. Show that if (x_n) converges to x in $C[a,b]$ in the uniform metric then (x_n) converges to x pointwise.

Let x_n in $C[0,1]$ be defined by

$$x_n(t) = \begin{cases} nt & \text{if } 0 \leq t \leq \frac{1}{n} \\ \frac{n(1-t)}{n-1} & \text{if } \frac{1}{n} \leq t \leq 1 \end{cases}$$

Sketch the graph of $x_n(t)$ and show that (x_n) converges pointwise to the function

$$x(t) = \begin{cases} 0 & \text{if } t = 0 \\ 1-t & \text{if } 0 < t \leq 1 \end{cases}$$

Deduce that (x_n) is not convergent in $C[0,1]$ in the uniform metric.

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Question 3. (a)[*7 marks*] Define what is meant by

(i) a *Cauchy sequence* in a metric space

(ii) a *complete metric space*.

Which of the following spaces X is complete and which is incomplete in the usual (Euclidean) metric? Give reasons.

$$X = \mathbb{Q}; \qquad X = \{\frac{1}{n} : n \in \mathbb{N}\}$$

(b) [*5 marks*] Let (X, d) be a metric space with the metric

$$d(x, y) = \begin{cases} 0 & \text{if} \quad x = y \\ 3 & \text{if} \quad x \neq y \end{cases}$$

Show that any Cauchy sequence in X is eventually constant, and deduce that (X, d) is complete.

(c) [*8 marks*] Explain what is meant by a *contraction* of a metric space. Show that if $f : [a, b] \to [a, b]$ is differentiable then f is a contraction if and only if there is a number $r < 1$ such that

$|Df(x)| \leq r$ for every $x \in (a, b)$.

State without proof the *Contraction Mapping Theorem*.

Show that the mapping $f : [-1, 1] \to [-1, 1]$ defined by

$$f(x) = \frac{1}{14}(3x^3 - 2x^2 + 9)$$

is a contraction, and deduce that there is a unique solution to the equation $3x^3 - 2x^2 - 14x + 9 = 0$ in the interval $[-1, 1]$.

Question 4. (a) [*6 marks*] Let (X, d) be a metric space and let $A \subseteq X$. What is meant by saying that A is *open*? Show that if $(A_i)_{i \in I}$ is any collection of open sets then the union $\bigcup_{i \in I} A_i$ is also open.

(b)[*8 marks*] What is meant by an *open ball* $B(a, r)$ in a metric space? Show that an open ball is open. By drawing a diagram or otherwise describe the open ball $B(a, 3)$ in $\mathbb{R}^2$, where $a = (3, 4)$

(i) with the usual metric;

(ii) with the max metric.

(c) [*6 marks*] Which of the following sets A is open in the given metric space X.

(i) $X = \mathbb{R}^2$ (with the Chicago metric); $A = \{(a, b) : a \geq b\}$

(ii) $X = C[0, 2]$ with the uniform metric; $A = \{x : x(0) < x(2)\}$

(iii) $X = \mathbb{R}$ with the usual metric; $A = \bigcup_{n \in \mathbb{Z}}(n, n + 1)$.

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Question 5. (a) [*10 marks*] Let $f : X \to Y$, where X and Y are metric spaces. Give the definition of f is *continuous* in terms of convergence of sequences. Show that if f is continous then

(i) if A is a closed subset of Y then $f^{-1}(A)$ is a closed subset of X ;

(ii) if $Y = X$ (so that $f : X \to X$) then $f^2 : X \to X$ is also continuous (where $f^2(x) = f(f(x))$).

(b) [*4 marks*] Suppose that $f, g : X \to \mathbb{R}$ are both continuous. Show that the function $h : X \to \mathbb{R}$ defined by

$$h(x) = 6f(x) - 5g(x)$$

is continuous.

(c) [*6 marks*] Let f be the function $f : C[0,1] \to \mathbb{R}$ defined for $x \in C[0,1]$ by

$$f(x) = x(0)$$

Show that f is not continuous with respect to the L_1 metric on $C[0,1]$ (and the usual metric on $\mathbb{R}$) by considering the functions $x_n(t)$ given by

$$x_n(t) = \begin{cases} (n-1)t & \text{if} \quad 0 \leq t \leq \dfrac{1}{n} \\ 1-t & \text{if} \quad \dfrac{1}{n} \leq t \leq 1 \end{cases}$$

(**Hint** Sketch the functions $x_n(t)$ and consider their limit in the L_1 metric.)

Question 6. (a) [*4 marks*] Let X be a metric space and $A \subseteq X$. What is meant by saying that (i) A is *bounded* and (ii) A is *compact*?

(b) [*3 marks*] Show that a compact set is closed;

(c) [*3 marks*] Show that a closed subset of a compact set is compact;

(d) [*4 marks*] Assuming that a closed bounded subset of $\mathbb{R}$ is compact, show that the same is true for $\mathbb{R}^2$;

(e) [*6 marks*] Which of the following sets is compact ?

(i)$\{(x,y) : -1 \leq x \leq y \leq 1\}$ in $\mathbb{R}^2$

(ii) $\{1, 1-\frac{1}{2}, 1-\frac{1}{3}, \ldots, \dfrac{n-1}{n}, \ldots\}$ in $\mathbb{R}$. Give reasons for your answers.

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Question 7. (a) [*3 marks*] Let X be a set and d_1 and d_2 be metrics on X. What is meant by saying that the metrics d_1 and d_2 are *equivalent*.

(b) [*5 marks*] Suppose that there are positive constants k, K such that

$$kd_1(x,y) \le d_2(x,y) \le Kd_1(x,y)$$

for all $x, y \in X$. Show that d_1 and d_2 are equivalent.

(c) [*4 marks*] Show that on $\mathbb{R}^2$ the usual (Euclidean) metric and the Chicago metric are equivalent.

(d) [*8 marks*] Suppose that d_1 and d_2 are equivalent metrics on a set X. Show that if $B_1(a,\varepsilon)$ is any open ball in the d_1 metric then there is an open ball $B_2(a,\delta)$ in the d_2 metric with $B_2(a,\delta) \subseteq B_1(a,\varepsilon)$. Deduce that if $x_n \to x$ in the d_2 metric then it is convergent in the d_1 metric.

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