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University of Swaziland

Supplementary Examination 2004/2005

B.Sc./B.Ed./B.A.S.S. IV

Title of Paper : **Partial Differential Equations**

Course Number : M 415

Time Allowed : Three (3) hours

Instructions :

1. This paper consists of **seven questions.**
2. Answer **any five questions**.
3. Your work must be accompanied by appropriate explanations.
4. Use of **cellular phones** during the examination is not allowed.
5. Only non-programmable calculators may be used.

Special requirements: None

The examination paper must not be opened until permission has been granted by the Invigilator.

Q1.
Determine the region in which the partial differential equation;

$$xu_{xx} + u_{yy} = x^2$$

is hyperbolic or parabolic. Transform the equation in the respective region to its canonical form. 20 [marks]

Q2.
Transform the equation $4u_{xx} + 5u_{xy} + u_{yy} + u_x + u_y = 0$ into canonical form and hence find its general solution. 20 [marks]

Q3.
Show that the solution to:

$$u_{tt}(x,t) = c^2 u_{xx}(x,t), -\infty < x < \infty, t > 0,$$

$$u(x,0) = f(x), u_t(x,0) = g(x), -\infty < x < \infty,$$

may be given in the *D'Alembert form*;

$$u(x,t) = \frac{1}{2}\{f(x+ct) + f(x-ct)\} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds.$$

20 [marks]

Q4.
The non-homogeneous wave equation:

$$u_{tt} - c^2 u_{xx} = h(x,t); -\infty < x < \infty, t > 0$$

$$u(x,0) = f(x), u_t(x,0) = g(x), -\infty < x < \infty,$$

has solution:

$$u(x,t) = \frac{1}{2}\{f(x+ct) + f(x-ct)\} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds + \frac{1}{2c}\int\int_{\Delta} h(x,t)d\Delta.$$

Solve this problem when:
(a)$h(x,t) = x + ct, u(x,0) = x$ and $u_t(x,0) = \sin x, -\infty < x < \infty.$
(b) $h(x,t) = e^x, f(x) = 2$ and $g = x^2.$ 20 [marks]

Q5.

The steady state temperature distribution $u(x,t)$ within a homogeneous semi-infinite rectangular plate satisfies:

$$u_{xx} + u_{yy} = 0, 0 < x < a, 0 < y < \infty,$$

$$u(x,0) = T_0, 0 \leq x \leq a,$$

$$u(0,y) = u(a,y) = 0, 0 < y < \infty.$$

Use the method of separation of variables to derive the solution:

$$u(x,y) = \frac{4T_0}{\pi} \sum_{n=1}^{\infty} \frac{e^{-\frac{(2n-1)\pi y}{a}}}{(2n-1)} \sin \frac{(2n-1)\pi x}{a}.$$

Hint: For $\sigma \geq 0$, where σ is the separation constant, there are no nontrivial solutions.

[20 marks]

Q6

(a) Consider the problem:

$$u_{tt} = c^2 u_{xx} + F(x), 0 < x < l, t > 0,$$

$$u(x,0) = f(x), u_t(x,0) = g(x), 0 \leq x \leq l,$$

$$u(0,t) = \alpha, u(l,t) = \beta, t \geq 0.$$

Set $u(x,t) = v(x,t) + U(x)$.

(a) Under what conditions does $v(x,t)$ satisfy the equation: $v_{tt} = c^2 v_{xx}$ with the appropriate homogeneous boundary conditions. It is known that this equation has solution:

$$v(x,t) = \sum_{n=1}^{\infty} \{a_n \cos \frac{n\pi ct}{l} + b_n \sin \frac{n\pi ct}{l}\} \sin \frac{n\pi x}{l}.$$

(b) Now use the above analysis to solve the following problem:

$$u_{tt} = c^2 u_{xx} + h, \text{ where } h \text{ is a constant},$$

$$u(x,0) = u_t(x,0) = 0, u(0,t) = u(l,t) = 0.$$

[20 marks]

Q7.

(a) Let $\mathcal{L}\{u(x,t)\} = U(x,s)$ be the Laplace transform of $u(x,t)$. Derive the following formulae:

1. $\mathcal{L}\{\frac{\partial u}{\partial t}\} = sU(x,s) - u(x,0)$
2. $\mathcal{L}\{\frac{\partial^2 u}{\partial t^2}\} = s^2U(x,s) - su(x,0) - u_t(x,0)$
3. $\mathcal{L}\{\frac{\partial u}{\partial x}\} = \frac{d}{dx}U(x,s)$
4. $\mathcal{L}\{\frac{\partial^2 u}{\partial x^2}\} = \frac{d^2}{dx^2}U(x,s)$.

(b)Use Laplace transforms to solve the equation:

$$u_t + xu_x = x, x > 0, t \geq 0,$$

$$u(x,0) = u(0,t) = 0, t > 0, x > 0.$$

20 [marks]

END OF PAPER