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UNIVERSITY OF SWAZILAND

SUPPLEMENTARY EXAMINATIONS 2005

BSc. / BEd. / B.A.S.S. III

TITLE OF PAPER	:	DYNAMICS II
COURSE NUMBER	:	M 355
TIME ALLOWED	:	THREE (3) HOURS
INSTRUCTIONS	:	1. THIS PAPER CONSISTS OF SEVEN QUESTIONS. 2. ANSWER ANY FIVE QUESTIONS
SPECIAL REQUIREMENTS	:	NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GRANTED BY THE INVIGILATOR.

QUESTION 1

1. Given that a mechanical system has transformation equations given by $\mathbf{r}_\nu = \mathbf{r}_\nu(q_1, q_2, \ldots, q_n)$ and kinetic energy given by $T = T(q_\alpha, \dot{q}_\alpha)$ where q_α are the generalized coordinates. Prove that

(a) $$\frac{\partial \dot{\mathbf{r}}_\nu}{\partial \dot{q}_\alpha} = \frac{\partial \mathbf{r}_\nu}{\partial q_\alpha}$$ [6 marks]

(b) $$2T = \sum_{\alpha=1}^{n} \dot{q}_\alpha \frac{\partial T}{\partial \dot{q}_\alpha}$$ [6 marks]

(c) Suppose that the potential V depends on q_α only. Prove that the quantity

$T + V$ is a constant

[8 marks]

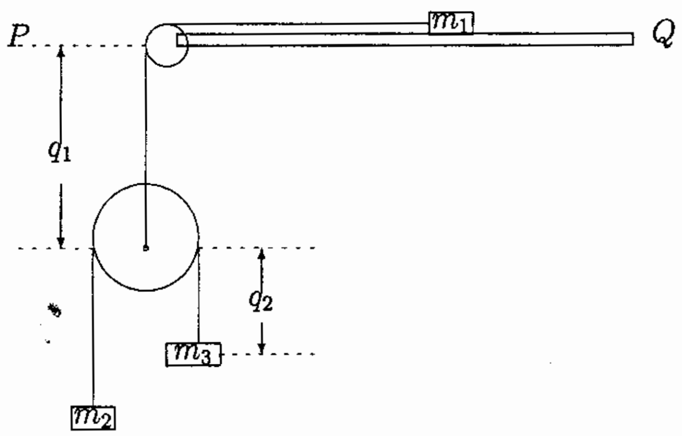

Figure 1:

QUESTION 2

2. Consider the pulley system shown in Figure (1) with generalized coordinates q_1 and q_2 as indicated. Assume that the pulleys have negligible masses and that

PQ is the reference level. Set up the Lagrangian for the system and prove that the Lagrange's equations of motion are given by

$$\begin{aligned}(m_1 + m_2 + m_3)\ddot{q}_1 + (m_3 - m_2)\ddot{q}_2 &= (m_3 + m_2)g \\ (m_3 - m_2)\ddot{q}_1 + (m_3 + m_2)\ddot{q}_2 &= (m_3 - m_2)g\end{aligned}$$

[20 marks]

QUESTION 3

3. (a) If the Hamiltonian

$$H = \sum_{\alpha=1}^{n} p_\alpha \dot{q}_\alpha - L$$

is expressed as a function of the generalised coordinates q_α and the momenta p_α ONLY and DOES NOT contain the time t explicitly, prove that

$$\dot{q}_\alpha = \frac{\partial H}{\partial p_\alpha} \quad , \qquad \dot{p}_\alpha = -\frac{\partial H}{\partial q_\alpha}$$

[8 marks]

(b) Given the following Lagrangian function

$$L = \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}m\dot{x}_2^2 - \frac{1}{2}\kappa(x_1^2 + x_2^2) - \frac{1}{2}\kappa(x_2 - x_1)^2$$

for a certain mechanical system.

i. Find the corresponding Hamiltonian function. [6 marks]

ii. Using Hamilton's equations obtain the equations of motion for the system. [6 marks]

QUESTION 4

4. (a) By the method of Poisson brackets, show that the transformation, given by

$$
\begin{aligned}
q_1 &= \sqrt{2P_1}\sin Q_1 + P_2, \qquad p_1 = \frac{1}{2}\left(\sqrt{2P_1}\cos Q_1 - Q_2\right) \\
q_2 &= \sqrt{2P_1}\cos Q_1 + Q_2, \qquad p_2 = -\frac{1}{2}\left(\sqrt{2P_1}\sin Q_1 - P_2\right)
\end{aligned}
$$

is canonical. [6 marks]

(b) For what values of the constant parameters α and β are the following transformations canonical

(i) $Q = q^\alpha \cos\beta p$, $P = q^\alpha \sin\beta p$ [4 marks]

(ii) $Q = q^\alpha e^{\beta p}$, $P = q^\alpha e^{-\beta p}$ [4 marks]

(c) Given that,

$$
\begin{aligned}
A_1 &= \frac{1}{4}(x^2 + p_x^2 - y^2 - p_y^2), \quad A_2 = \frac{1}{2}(xy + p_x p_y) \\
A_3 &= \frac{1}{2}(xp_y - yp_x), \quad A_4 = x^2 + y^2 + p_x^2 + p_y^2
\end{aligned}
$$

where x and y are generalized coordinates, and p_x and p_y are the generalized momenta, associated with x and y respectively. Show that

(i) $[A_1, A_2] = A_3$, (ii) $[A_3, A_2] = -A_1$ [3, 3 marks]

QUESTION 5

5. (a) If the Hamiltonian H is defined by the relation $H = \sum_{\alpha=1}^{n} p_\alpha \dot{q}_\alpha - L$ where H is the Hamiltonian and L is the Lagrangian, show that p_α and q_α are related by the equations

$$\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}, \quad \dot{q}_\alpha = \frac{\partial H}{\partial p_\alpha} \quad \text{and} \quad \dot{p}_\alpha = -\frac{\partial H}{\partial q_\alpha}$$

[8 marks]

(b) The Lagrangian function of a system is given by

$$L = \frac{1}{2}M\dot{R}^2 + \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}^2) - \frac{1}{2}k(r-b)^2$$

(i) Determine the cyclic (ignorable) coordinates. [2 marks]

(ii) Derive the Hamiltonian for the the system. [4 marks]

(iii) Using the Hamiltonian formulation show that the equation of motion corresponding to r is

$$\mu(\ddot{r} - r\dot{\theta}^2) + k(r-b) = 0$$

[6 marks]

QUESTION 6

6. (a) Show that the Euler-Lagrange equation for the functional $I = \int_a^b F(x, y, y', y'')dx$ is

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right) = 0$$

[10 Marks]

(b) Use the Euler-Lagrange equation given in (a) to find the extremal of the functional

$$I = \int_0^1 [(y'')^2 + y' + 3x^2]dx$$

subject to the boundary conditions $y(0) = 0$, $y(1) = 1$ and $y'(0) = 1$, $y'(1) = 1$ [10 Marks]

QUESTION 7

7. Find the curves that minimizes the following functionals subject to the given boundary conditions

(a) $\int_{-1}^{1} (x^2 y'^2 + 12y^2)dx, y(-1) = -1,\ y(1) = 1$ [10 marks]

(b) $\int_{x=1}^{e} (4y + xy'^2)dx$ if $y(e) = 0$, and $y(1)$ is not prescribed. [10 marks]