THE UNIVERSITY OF SWAZILAND

116

Department of Mathematics

Final Examination 2005

M331
REAL ANALYSIS

Three (3) hours

INSTRUCTIONS
1. This paper contains SEVEN questions.
2. Answer any FIVE questions.

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GRANTED BY THE INVIGILATOR.

Throughout this paper the symbols $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ stand for the natural numbers, the integers, the rational numbers and the real numbers respectively.

Question 1. Let A be a subset of the real numbers.

(a) [*10 marks*] What is meant by saying that A is *bounded*?

Which of the following sets is bounded? Give reasons for your answers.

(i) $\{q^{\frac{2}{3}} : q \in \mathbb{Q} \quad \text{and} \quad q < \sqrt{2}\}$

(ii) $\left\{\dfrac{6n^3+2}{n^3-6} : n \in \mathbb{N}\right\}$

(iii) $\{n^{-2} : n \in \mathbb{Z} \quad \text{and} \quad n \neq 0\}$

(b) [*6 marks*] What is meant by $\inf A$ for a set A that is bounded below?

Which of the sets in (a) is bounded below? Find $\inf A$ for each set that is bounded below.

(c) [*4 marks*] Which of the following statements is always true? Give a proof for those that are true and a counterexample for those that are false.

(i) if A is bounded below then $\mathbb{R} \setminus A$ is bounded above (where $\mathbb{R} \setminus A = \{x \in \mathbb{R} : x \notin A\}$);

(ii) if A is bounded below then the set $-A = \{-x : x \in A\}$ is bounded above.

Question 2. (a) [*8 marks*] Let (a_n) be a sequence of real numbers and $l \in \mathbb{R}$. Give a precise definition of the statement that

$$\lim_{n\to\infty} a_n = l$$

Show *directly from the definition* that

$$\lim_{n\to\infty} \frac{(1+2\sqrt{n})^2}{1+n} = 4.$$

(b) [*12 marks*] Which of the following sequences (a_n) is convergent? . For those that are, find the limit. State clearly any facts about limits that you use.

(i) $a_n = \dfrac{5n^2-n+2}{n-13n^4}$

(ii) $a_n = \dfrac{n-n^2+2n^3}{5n^3-17n}$

(iii) $a_n = \sqrt{2n^2 - \frac{1}{n^2}}$

(iv) $a_n = \sqrt{n+1} - \sqrt{n-1}$

CONT ...

Question 3. (a) [*4 marks*] Let (a_n) be a sequence. (i) Define what is meant by the *partial sums* of the series $\sum a_n$ (ii) What is meant by saying that $\sum_{n=1}^{\infty} a_n = s$

(b) [*6 marks*] Prove carefully that if $\sum_{n=1}^{\infty} a_n = s$ and $\sum_{n=1}^{\infty} b_n = t$ then $\sum_{n=1}^{\infty}(a_n + b_n) = s + t$.

(c) [*6 marks*] Show that each of the following series converges, stating any general theorems that you use.

(i) $\sum (-1)^n \frac{1}{n}$

(ii) $\sum \frac{n!}{n^n}$

(d) [*4 marks*] By considering $a_n = (-1)^n \frac{1}{\sqrt{n}} = b_n$ show that if $\sum a_n$ and $\sum b_n$ are both convergent then it does **not** necessarily follow that $\sum a_n b_n$ is convergent.

Question 4. (a) [*6 marks*] Find $\lim_{x \to c} f(x)$ for each of the following functions and the given value of c.

(i) $f(x) = \begin{cases} \dfrac{x^4 - 16}{x^2 - 4} & \text{if} \quad x \neq -2, 2 \\ 4 & \text{if} \quad x = -2 \text{ or } x = 2 \end{cases}$ and $c = -2$

(ii) $f(x) = \begin{cases} (x^2 - 1)\sin\left(\dfrac{1}{x-1}\right) & \text{if} \quad x \neq 1 \\ -1 & \text{if} \quad x = 1 \end{cases}$ and $c = 1$

(b) [*7 marks*] What is meant by saying that a function $f(x)$ is *continuous* at a point c. (You may assume that f is defined on an interval (a, b) that contains c.)

Prove that if $f(x)$ is continuous at c and $f(c) \neq 0$ then $\dfrac{1}{f(x)}$ is continuous at c.

(c) [*7 marks*] Which of the following functions is continuous at the given point c

(i) $f(x) = \begin{cases} \sqrt{x}\cos\dfrac{1}{x^2} & \text{if} \quad x \neq 0 \\ 0 & \text{if} \quad x = 0 \end{cases}$; $c = 0$

(ii) $f(x) = [\sqrt{x}]$ (where $[x]$ is the integer part of x - that is, the greatest integer $\leq x$); $c = 9$

(Give reasons for your answers.)

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Question 5. Let $f : [a,b] \to \mathbb{R}$ and let $a < c < b$.

(a) [*8 marks*] Define what is meant by saying that f is *differentiable* at the point c. Show carefully that if f and g are differentiable at c then $f.g$ is differentiable at c. (You may assume without proof that if f is differentiable at c then f is continuous at c.)

(b) [*6 marks*] Prove that if f is differentiable at c and c is a local maximum or local minimum of f, then $Df(c) = 0$. Give an example to show that the converse is false.

(c) [*6 marks*] State the Mean Value Theorem and use it to show that

$$|\cos^4 x - \cos^4 y| \le 4|x-y|$$

for all $x, y \in \mathbb{R}$.

Question 6. (a) [*10 marks*] Let $f : [a,b] \to \mathbb{R}$ be any function. Define what is meant by saying that f is *Riemann integrable* using upper and lower sums, and what is meant by the *Riemann integral* $\int_a^b f(x)dx$.

Explain briefly why f is Riemann integrable if it is continuous.

(b) [*10 marks*] State the *integral test* for convergence of a series and sketch its proof, stating clearly any property of the Riemann integral that you use.

Question 7. (a) [*10 marks*] Suppose that $f : [a,b] \to \mathbb{R}$ is continuous and $F : [a,b] \to \mathbb{R}$ is differentiable with $DF(x) = f(x)$ for $a \le x \le b$. Show that

$$\int_a^x f(t)dt = F(x) - F(a)$$

for each $x \in [a,b]$. (You may assume that the function $G(x) = \int_a^x f(t)dt$ is differentiable with derivative $DG(x) = f(x)$; state carefully any other properties of derivatives or integrals that you assume.)

(b) [*10 marks*] Let $g(x) = \int_0^{x^2} \exp(1 + \sin(t^2))\, dt$. Show that g is differentiable for all x and find its derivative.

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