UNIVERSITY OF SWAZILAND

FINAL EXAMINATIONS 2005

B.Sc. / B.Ed. / B.A.S.S. II

TITLE OF PAPER	:	FOUNDATIONS OF MATHEMATICS
COURSE NUMBER	:	M231
TIME ALLOWED	:	THREE (3) HOURS
INSTRUCTIONS	:	1. THIS PAPER CONSISTS OF SEVEN QUESTIONS. 2. ANSWER ANY FIVE QUESTIONS
SPECIAL REQUIREMENTS	:	NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GRANTED BY THE INVIGILATOR.

QUESTION 1

(a) State the difference between deductive reasoning and inductive reasoning. Which of the two is a valid form of argument? Explain. [4]

(b) Prove that if n is an integer and n^2 is divisible by 2, then so is n. [6]

(c) Using the result in part (b), or otherwise, prove that if r is a real number such that $r^2 = 2$, then r is irrational. [10]

QUESTION 2

(a) Write the negation of the following statement: "The function f of one variable is *bounded above* if and only if there is a real number y such that for every real number x, $f(x) \leq y$." [5]

(b) Show that if $x > 2$ is a real number, then there is a unique real number $y < 0$ such that $x = \dfrac{4y}{2+2y}$. [6]

(c) Critic Ivor Smallbrain has been thrown into prison for libelling the great film director Michael Loser. During one of his needlework classes in prison, Ivor is given a pile of pieces of leather in the shapes of regular pentagons and regular hexagons, and is told to sew some of these together into a convex polyhedron (which will then be used as a soccer ball). He is told that each vertex must lie on exactly 3 edges (that is, the corresponding plane graph is regular of degree 3). Ivor immediately exclaims, "Then I need exactly 12 pentagonal pieces."

Prove that Ivor is correct. [9]

QUESTION 3

(a) Prove that if n points are placed on the circumference of a circle and chords are drawn from each point to all other points in such a way that no three chords have a common point of intersection, then the circle is divided into $\binom{n}{4} + \binom{n}{2} + 1$ regions. [10]

(b) Suppose n rings, with different outside diameters, are slipped onto an upright peg, the largest ring at the bottom, the second largest on top of it, and so on, so that the smallest ring is at the top, to form a pyramid. Two other upright pegs are placed sufficiently far apart. We wish to transfer all the rings, one at a time, to the second peg to form an identical pyramid. During the transfers, we are not permitted to place a larger ring on top of a smaller one (which necessitates the third peg). What is the smallest number of moves necessary to complete the transfer? [10]

QUESTION 4

(a) The *Fibonacci sequence* is a sequence of integers $u_1, u_2, \ldots, u_n, u_{n+1}, \ldots$, such that $u_1 = 1$, $u_2 = 1$ and

$$u_{n+1} = u_n + u_{n-1}$$

for all $n \geq 1$. The beginning of this sequence is $1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \ldots$. Prove by strong induction that for all positive integers n,

$$u_n = \frac{1}{\sqrt{5}}(\alpha^n - \beta^n),$$

$$\text{where} \qquad \alpha = \frac{1+\sqrt{5}}{2} \qquad \text{and} \qquad \beta = \frac{1-\sqrt{5}}{2}$$

[12]

(b) Suppose that Canada Post prints only 3 cent and 5 cent stamps. Prove that it is possible to make up any postage of n cents using only 3 cent and 5 cent stamps for $n \geq 8$. [8]

QUESTION 5

(a) Let $x = 0.a_1a_2a_3\ldots$, where for $n = 1, 2, 3, \ldots$, the value of a_n is the number 0, or 1, or 2, or 3 which is the remainder on dividing n by 4. Is x rational? If so, express x as a fraction $\frac{m}{n}$ where m and n are integers with $n \neq 0$. [8]

(b) Prove that between any two different irrational numbers there is a rational number and an irrational number. [12]

QUESTION 6

(a) (i) Define an equivalence relation. [2]

(ii) Show that the relation

$$\mathcal{R} = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} : x \equiv y \pmod{2}\}$$

is an equivalence relation. What are the equivalence classes of $\mathcal{R}$? [12]

(b) (i) Define the composition $f \circ g$ of any two functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ and $g : \mathbb{R} \longrightarrow \mathbb{R}$. [2]

(ii) Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ and $g : \mathbb{R} \longrightarrow \mathbb{R}$ be the functions defined by $f(x) = \sin x$ and $g(x) = x^2 + 2$ for all $x \in \mathbb{R}$. Determine $(f \circ g)(x)$ and $(g \circ f)(x)$. [4]

QUESTION 7

(a) State and prove the Fundamental Theorem of Arithmetic. [12]

(b) Prove that there are infinitely many primes of the form $3k + 2$, where k is an integer. [8]

END OF EXAMINATION